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A Two-Parameter Inverted Exponentiated Skewed Student-T Distribution: Theory and Application

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ABSTRACT

In this study, a new two-parameter generalization of the skew-t distribution called the Inverted Exponentiated Skew t distribution, has been proposed. Some properties of the model like order statistics, entropy, asymptotic behavior, moments, characteristic function, and quantile function were derived. Parameter estimates using maximum likelihood estimation were consistent and the behavior of the estimates with increase in sample size was studied using Montecarlo simulation. The flexibility of the Inverted Exponentiated Skew Student t model was demonstrated by application to four financial datasets and the results revealed that the Inverted Exponentiated Skew Student t model yielded a better fit.

1. Introduction

The study on the flexibility of continuous probability distributions has been a central theme in distribution theory literature, driven by the limitations of classical models in handling diverse datasets which are the results of hybrid/complex data generating process in the 21st century (Carta *et al.*, 2009). In financial data for example, distinctive stylized features such as heavy-tail, extreme events that defy conventional classical assumptions, high skewness and kurtosis, highlighting asymmetry and fat tails in financial returns etc. have assumed prominence, introducing challenges and opportunities in the modeling (Malmsten and Teräsvirta 2010). As financial markets continue to evolve, the need for hybrid probability distributions becomes increasingly evident, as they offer the requisite flexibility to accommodate these stylized features and fortify the resilience of financial modeling efforts.

One of the classical models that has found several applications in financial time series analysis is the student t distribution derived by Willieam S. Gosset (1908). Blattberg and Gonedes (1974) assessed the fitness of the independent *t* distributions for stock return data, and Zellner (1976) analyzed stock prices. Recent applications were also found in risk management using Chilean stock markets (Cademartori *et al.*, 2003), European option pricing (Cassidy *et al.*, 2010), insurance loss data models (Brazauskas and Kleefeld, 2011), Brazilian stock return modeling (Bergmann and de Oliveira, 2013), and Nigerian stock return modeling (OI *et al.*, 2014). The student *t* model has evolved over the years and have resulted in various approximations and generalizations of the probability model (see Li and Nadarajah 2020).

In recent times, several successes have been reported in distribution theory literature on method (family of distributions) utilized in hybridizing classical probability models. among the methods are; Gupta *et al.*, (1998) proposed the exponentiated family of distribution. Cordeiro *et al.*, (2013) derived the Exponentiated Generalized-G family of distributions. Tahir *et al.*, (2015) introduced the New Weibull-G, and Yousof *et al.*, (2017) proposed the Burr X-G family. Also, Alizadeh *et al.*, (2017) derived the Generalized Odd Generalized Exponential Family, and Hamedani *et al.*, (2018) introduced the Type I General Exponential Class of Distributions. Alizadeh *et al.*, (2018) presented the Transmuted Weibull-G family, Mansour *et al.*, (2019) introduced the Transmuted Transmuted-G family. This diverse collection of distributions highlights the ongoing efforts of researchers to enhance the flexibility and applicability of probability models.

By applying one or more of the above methods, the student t distribution has been modified and hybridized to improve its flexibility and application. Aas and Haff (2006) proposed the generalized hyperbolic skew-t (GHST)

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distribution, Roozegar *et al.*, (2016) proposed the t skewed t distribution, Dikko *et al* (2017) worked on the exponentiated skew t distribution, Said *et al.*, (2017) proposed the Kumaraswamy skew-t distribution, Basalamah *et al.*, (2018) derived the Beta skew-t (BST) distribution, Adubisi *et al*., (2021) derived the Type I Half Logistic Skew-t Distribution.

This article focuses on proposing a new family of distribution called the Inverted Exponentiated (IE) family of distribution and use the generator in deriving a sub model called the Inverted Exponentiated Skew Student *t* distribution so as improve the flexibility of the skew *t* and its application to financial data. This will generate a heavy-tailed distribution with enhanced flexibility, and he ability to capture skewness and kurtosis and with a unimodal characteristic. Such a distribution is valuable for its adaptability across different analytical scenarios. The rest of the article is structured into six sections, viz: Section 2 introduces the IE family of distribution and the Inverted Exponentiated Skew Student *t* (IESS*t*) distribution. Section 3 outlines its statistical properties. Section 4 discusses parameter estimates using maximum likelihood estimation and a Montecarlo simulation study. Section 5 demonstrates the usefulness of IESS*t* distribution with four financial datasets, and Section 6 concludes.

2 Methods

2.1 Inverted Exponentiated Family of Distribution

2.1.1 Exponentiated family of distribution

Gupta *et al.*, (1998) introduced a family of Exponentiated distributions based on cumulative distribution function CDF of the baseline distribution.

$$F(y) = G(y)^u; \text{ for } u > 0 \tag{1}$$

where $G(y)$ is the CDF of any baseline distributions. To obtain the PDF of the above CDF is by taking differentiated with respect to y to give;

$$f(y) = uG(y)^{u-1} g(y) \tag{2}$$

where $u > 0$, the shape parameter and $g(y) = \dfrac{dG(y)}{dy}$

2.1.2 Inverting a probability density function

The procedure of inverting the density function used in this study was adapted from that described by Chesneau *et al.,*, (2020). The CDF of the exponentiated family distribution is given by (1). Let $F(y) = y$ and $G(y) = x$. Then it implies that $F(y) = x^u = y$, $y^{\frac{1}{u}} = x$ and $F^{-1}(y) = y^{\frac{1}{u}}$, thus implying that

$$F(y) = G(y)^{\frac{1}{u}} \tag{3}$$

which is the inverted CDF of the Inverted Exponentiated Distribution (IED).

2.1.3 Inverted Exponentiated Skewed Student-*t* Distribution (IESStD)

The Inverted Exponentiated generator as derived and given by the expression in (3) and the CDF of the Skewed Student-t Distribution is given as;

$$G(y) = \left[\frac{1}{2} + \frac{y}{2\sqrt{\lambda + y^2}}\right]; -\infty < y < \infty \text{ ; (Jones, and Faddy, 2007)} \tag{4}$$

Substituting (4) into equation (3), the CDF of IESStD is obtained as;

$$F(y)=\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1}{u}} \tag{5}$$

By the fundamental theory of calculus, the PDF of the IESStD is obtained from the CDF by differentiating with respect to (wrt) the random variable *y*, given as;

$$f(y)=\frac{d}{dy}F(y)$$

$$f(y)=\frac{d}{dy}\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1}{u}}$$

$$f(y)=\frac{1}{u}\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1}{u}-1}\frac{2\lambda}{4\left(\lambda+y^2\right)^{\frac{3}{2}}}$$

Therefore;

$y\sim IESStD(u,\lambda)$, its PDF is given by;

$$f(y)=\frac{\lambda}{2u\left(\lambda+y^2\right)^{\frac{3}{2}}}\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1-u}{u}} \tag{6}$$

For $-\infty<y<\infty$

Let *y* be a random variable from an arbitrary baseline distribution. Hence, *y* is said to follow the proposed Inverted Exponentiated Skewed student-t distribution (IESStD) if it has its CDF and PDF as presented in equation (5) and (6).

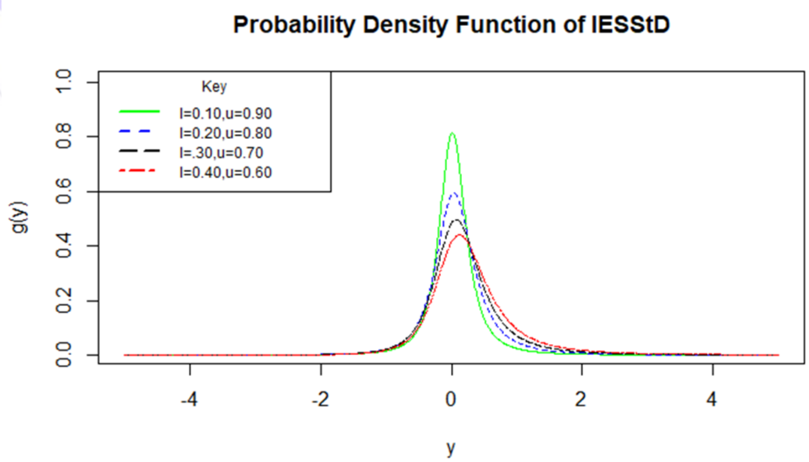

Figure 1: Probability Density Function plot of IESStD at different parameter values where $l=\lambda$

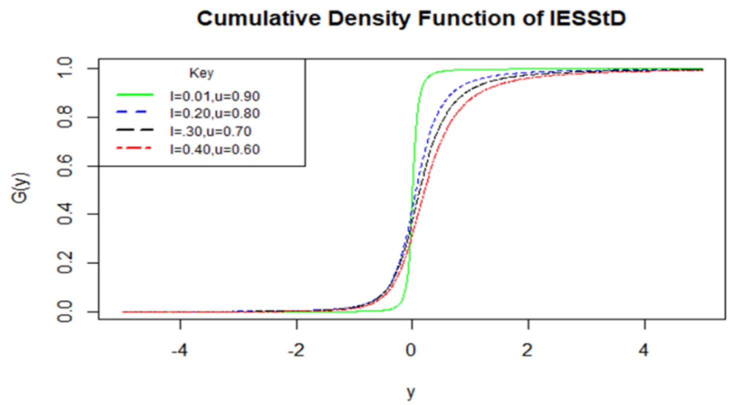

Figure 2: Cumulative Density Function plot of IESStD at Different Parameter values where $l = \lambda$

In Figure 1, the PDF plot of the IESStD reveals crucial insights into the distribution's shape and likelihood of different values. Asymmetry is evident with a prolonged tail on one side and a shorter, steeper tail on the other. The peak of the curve signifies the mode or the most probable value, while the heavier tails, characteristic of the IESStD, suggest a greater probability of extreme values. In Figure 2, the CDF plot of the IESStD. The gradual increase from 0 to 1 (on the y axis) indicates the increasing likelihood of higher values, with any changes in slope corresponding to critical points. The point where the CDF reaches 0.5 marks the median of the distribution, highlighting its central tendency.

2.1.4 Asymptotic behaviour of IESStD

Proposition 1

The limit of Inverted Exponentiated Skewed student-t density function as $y \to \infty$ is 0 and the limit as $y \to -\infty$ is 0.

Proof:

These can be shown by getting the limit of the Inverted Exponentiated Skewed student-t density function

For $y \to \infty$:

$$\lim_{y\to\infty} g(y) = \lim_{z_t\to\infty} \left\{ \frac{1}{u} \left[\frac{1}{2} + \frac{y}{2\sqrt{\lambda + y^2}} \right]^{\frac{1-u}{u}} \frac{2\lambda}{4\left(\lambda + y^2\right)^{\frac{3}{2}}} \right\} \tag{7}$$

$$= \lim_{y\to\infty} \left\{ \frac{1}{u} \left[\frac{1}{2} + \frac{y}{2\sqrt{\lambda + y^2}} \right]^{\frac{1-u}{u}} \frac{\lambda}{2\left(\lambda + y^2\right)^{\frac{3}{2}}} \right\} \tag{8}$$

Since $\lim_{y\to\infty}\left(\frac{\lambda}{2\left(\lambda+y^2\right)^{\frac{3}{2}}}\right)=0$

Then

$$\lim_{y\to\infty} g\left(y\right)=\lim_{z_t\to\infty}\left\{\frac{1}{u}\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1-u}{u}}\frac{2\lambda}{4\left(\lambda+y^2\right)^{\frac{3}{2}}}\right\}=0 \tag{9}$$

and let $y\to-\infty$

$$\lim_{y\to-\infty} g\left(y\right)=\lim_{z_t\to\infty}\left\{\frac{1}{u}\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1-u}{u}}\frac{2\lambda}{4\left(\lambda+y^2\right)^{\frac{3}{2}}}\right\}$$

$$=\lim_{y\to-\infty}\left\{\frac{1}{u}\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1-u}{u}}\frac{\lambda}{2\left(\lambda+y^2\right)^{\frac{3}{2}}}\right\} \tag{10}$$

Since $\lim_{y\to-\infty}\left(\frac{\lambda}{2\left(\lambda+y^2\right)^{\frac{3}{2}}}\right)=0$, Then

$$\lim_{y\to-\infty} g\left(y\right)=\lim_{z_t\to-\infty}\left\{\frac{1}{u}\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1-u}{u}}\frac{2\lambda}{4\left(\lambda+y^2\right)^{\frac{3}{2}}}\right\}=0 \tag{11}$$

This is an indication that the proposed IESStD is unimodal.

Properties of IESS*t*D

Binomial Expansion of the Proposed Probability Distributions

In the following, the PDF and CDF of the proposed IESStD was we expand using series expansion in order to easily compute the some of the properties of the distribution. Thus, if “a” is a positive real non-integer and $|y| \leq 1$, we can consider the power series expansion given as;

$$\left(1-y\right)^{b-1}=\sum_{i=0}^{\infty}b_i y^i \tag{12}$$

where $b_i=\left(-1\right)^i\binom{b-1}{i}$; and also the sum of the binomial series given by;

$$\left(a+b\right)^n=\sum_{i=0}^{\infty}\binom{n}{k}\left(a^{n-k}b^k\right) \tag{13}$$

Binomial expansion of PDF of IESStD

The PDF of IESStD, is given as;

$$g\left(y\right)=\frac{\lambda}{2u}\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1-u}{u}}.\frac{1}{\left(\lambda+y^2\right)^{\frac{3}{2}}}=\frac{\lambda}{2u}\left[\left(\frac{1}{2}+\frac{y}{\sqrt{\lambda+y^2}}\right)\right]^{\frac{1-u}{u}}.\frac{1}{\left(\lambda+y^2\right)^{\frac{3}{2}}} \tag{14}$$

$$=\frac{\lambda}{2u}\left[\frac{1}{2}\right]^{\frac{1-u}{u}}\left[\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right]^{\frac{1-u}{u}}.\frac{1}{\left(\lambda+y^2\right)^{\frac{3}{2}}}$$

$$=\frac{2\lambda}{2.2^{\frac{1}{u}}.u}\left[\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)\right]^{\frac{1-u}{u}}.\frac{1}{\left(\lambda+y^2\right)^{\frac{3}{2}}}$$

$$=\frac{\lambda}{2^{\frac{1}{u}}.u}\left[1+\frac{y}{\sqrt{\lambda+y^2}}\right]^{\frac{1-u}{u}}.\frac{1}{\left(\lambda+y^2\right)^{\frac{3}{2}}} \tag{15}$$

From (15), let $a=1, b=\frac{y}{\sqrt{\lambda+y^2}}$, then $n=\frac{1}{u}-1$

On substituting and simplifying, led to the following;

$$\left(a+b\right)^n=\left(1+\frac{y}{\sqrt{\lambda+y^2}}\right)^{\frac{1}{u}-1}=\sum_{i=0}^{\frac{1}{u}-1}\binom{\frac{1}{u}-1}{i}\left(1\right)^{\left(\frac{1}{u}-1\right)-i}\left[\frac{y}{\sqrt{\lambda+y^2}}\right]^i \tag{16}$$

Thus, substituting (16) into (15), we have

$$=\frac{\lambda}{2^{\frac{1}{u}}.u}\sum_{i=0}^{\frac{1}{u}-1}\binom{\frac{1}{u}-1}{i}\left[\frac{y}{\sqrt{\lambda+y^2}}\right]^i.\frac{1}{\left(\lambda+y^2\right)^{\frac{3}{2}}}$$

$$= \frac{\lambda}{2^{\frac{1}{u}}.u} \sum_{i=0}^{\frac{1}{u}-1} \binom{\frac{1}{u}-1}{i} y^i \left[\lambda + y^2\right]^{-\frac{i}{2}} . \left(\lambda + y^2\right)^{-\frac{3}{2}}$$

$$g(y) = \frac{\lambda}{2^{\frac{1}{u}}.u} \sum_{i=0}^{\frac{1}{u}-1} \binom{\frac{1}{u}-1}{i} y^i \left[\lambda + y^2\right]^{-\left(\frac{3+i}{2}\right)} \tag{17}$$

$$g(y) = h_{u,\lambda} y^i \left[\lambda + y^2\right]^{-\left(\frac{3+i}{2}\right)} \tag{18}$$

where; $h_{u,\lambda} = \frac{\lambda}{2^{\frac{1}{u}}.u} \sum_{i=0}^{\frac{1}{u}-1} \binom{\frac{1}{u}-1}{i}$

Binomial expansion of CDF of IESStD

Using (13) the CDF of IESStD can be represented as a sum of series as follows.

$$G(y) = \left[\frac{1}{2} + \frac{y}{2\sqrt{\lambda + y^2}}\right]^{\frac{1}{u}} = \sum_{k=0}^{\infty} \binom{\frac{1}{u}}{k} \left(\frac{1}{2}\right)^{\frac{1}{u}-k} \left(\frac{y}{2\sqrt{\lambda + y^2}}\right)^k \tag{19}$$

Let $q = \sum_{k=0}^{\infty} \binom{\frac{1}{u}}{k} \left(\frac{1}{2}\right)^{\frac{1}{u}}$, then;

$$G(y) = q\left(\frac{y}{2\sqrt{\lambda + y^2}}\right)^k = qy^k \left(\lambda + y^2\right)^{-\frac{k}{2}} \tag{20}$$

Quantile function of IESStD

The corresponding quantile function $Q(p), 0 < p < 1$ for the Inverted Exponentiated Skewed Student-*t* model is derived as follows;

let

$$Q(y) = G(y) \tag{21}$$

where $G(y)$ is the CDF of the density function. Then,

$$\Phi(\mu) = G^{-1}(\mu)$$

Where; $\mu \sim Uniform(0,1)$.

If $G(y)$ is given as $G(y)=\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1}{u}}$, and

$\mu=G(y)$, then (22)

$$\mu=\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1}{u}}$$

$$\mu^u=\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right];\Rightarrow 2\mu^u=1+\frac{y}{\sqrt{\lambda+y^2}};\Rightarrow 2\mu^u-1=\frac{y}{\sqrt{\lambda+y^2}}$$

$$\left(\sqrt{\lambda+y^2}\right)\left(2\mu^u-1\right)=y$$

$$\left(\lambda+y^2\right)\left(2\mu^u-1\right)^2=y^2$$

$$\frac{\lambda\left(2\mu^u-1\right)^2}{\left[1-\left(2\mu^u-1\right)^2\right]}=y^2$$

$$\Phi(\mu)=\pm\sqrt{\frac{\lambda\left(2\mu^u-1\right)^2}{\left[1-\left(2\mu^u-1\right)^2\right]}} \tag{23}$$

Moment function of IESStD

By definition, the r^{th} moment of a continuous random variable y is given by;

$$u_r=E\left[y^r\right]=\int_{-\infty}^{\infty}y^r f(y)dy \tag{24}$$

Let $y\sim IESStD$, then the r^{th} moment of the random variable is derived as follows. On substituting (18) into(24) , we have that

$$u_r=\int_{-\infty}^{\infty}y^r h_{u,\lambda}y^i\left[\lambda+y^2\right]^{-\left(\frac{3+i}{2}\right)}dy \tag{25}$$

$$=h_{u,\lambda}(\lambda)^{-\left(\frac{3+i}{2}\right)}\int_{-\infty}^{\infty}y^r y^i\left[1+\frac{y^2}{\lambda}\right]^{-\left(\frac{3+i}{2}\right)}dy$$

$$=h_{u,\lambda}(\lambda)^{-\left(\frac{3+i}{2}\right)}\int_{-\infty}^{\infty}y^{r+i}\left[1+\frac{y^2}{\lambda}\right]^{-\left(\frac{3+i}{2}\right)}dy \tag{26}$$

By transformation, let $p=\frac{y^2}{\lambda};p\lambda=y^2;y=(p\lambda)^{\frac{1}{2}};\frac{dp}{dy}=\frac{2}{\lambda}y;dy=\frac{\lambda}{2y}dp$ (27)

Substitute (27) into (26)

$$=h_{u,\lambda}\left(\lambda\right)^{-\left(\frac{3+i}{2}\right)}\int_{-\infty}^{\infty}y^{r+i}\left[1+p\right]^{-\left(\frac{3+i}{2}\right)}.\frac{\lambda}{2y}dp$$

$$=h_{u,\lambda}\lambda^{\frac{r+i-1}{2}}\left(\lambda\right)^{-\left(\frac{3+i}{2}\right)+1}\int_{0}^{\infty}\left(p\right)^{\frac{r+i-1}{2}}\left[1+p\right]^{-\left(\frac{3+i}{2}\right)}dp$$

$$u_r=h_{u,\lambda}\lambda^{\frac{r-2}{2}}\int_{0}^{\infty}\frac{\left(p\right)^{\frac{r+i-1}{2}}}{\left[1+p\right]^{\left(\frac{3+i}{2}\right)}}dp \tag{28}$$

Recall that

$$B\left[p,q\right]=\int_{0}^{\infty}\frac{y^{p-1}}{\left(1+y\right)^{p+q}}dy \tag{29}$$

Therefore,

$$u_r=h_{u,\lambda}\lambda^{\frac{r-2}{2}}\int_{0}^{\infty}\frac{\left(p\right)^{\frac{r+i-1}{2}}}{\left[1+p\right]^{\left(\frac{3+i}{2}\right)}}dp \tag{30}$$

Note:

$$\frac{r+i-1}{2}=p-1;\frac{r+i-1}{2}+1=p;\Rightarrow\frac{r+i-1+2}{2}=p;\Rightarrow\frac{r+i+1}{2}=p$$

Also, $p+q=\frac{3+i}{2}\Rightarrow\frac{r+i+1}{2}+q=\frac{3+i}{2}\Rightarrow\frac{3+i}{2}-\frac{r+i+1}{2}=q;\Rightarrow\frac{3+i-r-i-1}{2}\Rightarrow\frac{2-r}{2}=q$

Equation (30) becomes; $u_r=h_{u,\lambda}\lambda^{\frac{r-2}{2}}\int_{0}^{\infty}\frac{\left(p\right)^{\frac{r+i-1}{2}-1}}{\left[1+p\right]^{\left(\frac{3+i}{2}\right)+\left[\frac{2-r}{2}\right]}}dp$

$$u_r=h_{u,\lambda}\lambda^{\frac{r-2}{2}}B\left[\frac{r+i-1}{2},\frac{2-r}{2}\right]$$

Hence the moment of the IESStD is given as;

$$u_{rg}=\frac{\lambda}{2^{\frac{1}{u}}.u}\sum_{i=0}^{\frac{1}{u}-1}\binom{\frac{1}{u}-1}{i}\lambda^{\frac{r-2}{2}}\beta\left[\frac{r+i-1}{2},\frac{2-r}{2}\right] \tag{31}$$

Characteristic function of IESStD

The characteristic function for a random variable *y* with PDF $g\left(y\right)$ is defined for $y\in \boldsymbol{R}$ as;

$$\varphi_y\left(t\right)=\mathrm{E}\left[e^{itz_t}\right]=\int_{-\infty}^{\infty}e^{ity}g\left(y\right)dy \tag{32}$$

On substituting equation (18) into (32) , the CF of the IESStD is obtained as follow:

$$\varphi_y(t) = E\left[e^{ity}\right] = \int_{-\infty}^{\infty} e^{ity} h_{u,\lambda} y^i \left[\lambda + y^2\right]^{-\left(\frac{3+i}{2}\right)} dy \tag{33}$$

Recall from Taylor's series expansion that;

$$e^{ity} = \sum_{0}^{\infty} \frac{(it)^r}{r!} y^r \tag{34}$$

Substitute (34) into (33)

$$= \int_{-\infty}^{\infty} \left(\sum_{0}^{\infty} \frac{(it)^r}{r!} y^r \right) h_{u,\lambda} y^i \left[\lambda + y^2\right]^{-\left(\frac{3+i}{2}\right)} dy$$

$$= h_{u,\lambda} \sum_{y=0}^{\infty} \frac{(it)^r}{r!} \int_{-\infty}^{\infty} y^{r+i} \left[\lambda + y^2\right]^{-\left(\frac{3+i}{2}\right)} dy \tag{35}$$

Recall from the moment expression in (31), substituting into (35) yields the characteristic function given as;

$$\varphi_y(t) = \sum_{y=0}^{\infty} \frac{(it)^r}{r!} h_{u,\lambda} \lambda^{\frac{r-2}{2}} B\left[\frac{r+i-1}{2}, \frac{2-r}{2}\right] \tag{36}$$

Order Statistic

Order statistics are the sorted values of a set of random variables. These statistics are useful in statistical inference, providing information about the distribution and characteristics of the underlying population (Dey, Raheem, and Mukherjee 2017). Let $Y_1 < Y_2 < \ldots < Y_c$ be an ordered random sample from a valid continuous density function, then, the c^{th} order statistic is given as;

$$f_{(c,n)}(y) = \frac{n!}{(c-1)!(n-c)!} g(y)\left[G(y)\right]^{c-1}\left[1-G(y)\right]^{n-c} \tag{37}$$

Expanding $\left[1-G(y)\right]^{n-c}$ using the Binomial theorem, we have

$$\left[1-G(y)\right]^{n-c} = \sum_{p=0}^{\infty} \binom{n-c}{p} (-1)^p \left[G(y)\right]^p \tag{38}$$

Therefore, equation (37) becomes

$$f_{(c,n)}(y) = \sum_{p=0}^{\infty} \binom{n-c}{p} \frac{n!}{(c-1)!(n-c)!} (-1)^p g(y)\left[G(y)\right]^{p+c-1} \tag{39}$$

The above expression in (39) will be used to derive the smallest and largest order statistic for the proposed probability model, and to do that, we will utilize the sum of the binomial series of for the PDF and CDF of the distributions in (40) and (41) for IESStD.

Order statistic for IESStD

To obtain the c^{th} order statistic for the IESStD, we make the substitution (42) for $g(y)$ and (43)for $G(y)$ in equation (39) to obtain;

$$f_{(c,n)IE}(y) = \sum_{p=0}^{\infty} \binom{n-c}{p} \frac{n!}{(c-1)!(n-c)!} (-1)^p h_{u,\lambda} y^i \left[\lambda + y^2\right]^{-\left(\frac{3+i}{2}\right)} \left[q 2^{-k} y^k \left(\lambda + y^2\right)^{-\frac{k}{2}}\right]^{p+c-1}$$

$$f_{(c,n)IE}(y) = \sum_{p=0}^{\infty} \binom{n-c}{p} \frac{n!}{(c-1)!(n-c)!} (-1)^p h_{u,\lambda} q^{p+c-1} y^{k(p+c-1)+i} \left[\lambda + y^2\right]^{-\left(\frac{3+i}{2}\right)-\left(\frac{k}{2}\right)p+c-1}$$

The c^{th} order statistic for the IESStD is thus expressed as;

$$f_{(c,n)IE}(y) = h_{(c,n)IE} y^{k(p+c-1)+i} \left[\lambda + y^2\right]^{-\left(\frac{3+i}{2}\right)-\left(\frac{k}{2}\right)p+c-1} \tag{44}$$

Where $h_{(c,n)IE} = \sum_{p=0}^{\infty} \binom{n-c}{p} \frac{n!}{(c-1)!(n-c)!} (-1)^p h_{u,\lambda} q^{p+c-1}$

The smallest and largest order statistic for the IESStD will be obtained by substituting $c=1$ and $c=n$ in equation (44).

Smallest order statistic

$$f_{(1,n)IE}(y) = h_{(1,n)IE} y^{k(p+1-1)+i} \left[\lambda + y^2\right]^{-\left(\frac{3+i}{2}\right)-\left(\frac{k}{2}\right)p+1-1}$$

$$f_{(c,1)IE}(y) = h_{(c,1)IE} y^{k(p)+i} \left[\lambda + y^2\right]^{-\left(\frac{3+i}{2}\right)-\left(\frac{k}{2}\right)p} \tag{45}$$

Largest order statistic

$$f_{(n,n)IE}(y) = h_{(c,n)IE} y^{k(p+n-1)+i} \left[\lambda + y^2\right]^{-\left(\frac{3+i}{2}\right)-\left(\frac{k}{2}\right)p+n-1} \tag{46}$$

Reliability Functions

Reliability analysis in distribution theory is basically concerned with the analysis of time to failure for a system under study and in this section, the reliability measures/functions for the proposed distributions will be derived.

Survival (*S*(*y*)) and cumulative hazard (*H*(*y*)) function

One of the functions usually estimated in reliability analysis is the survival function. The survival function is defined as the probability that the time to the event is greater than a specified time *t*. This is obtained by subtracting 1 from the CDF of a distribution. Mathematically, it can be expressed as;

$$S(y) = 1 - G(y) \tag{47}$$

The hazard function, represents the instantaneous rate of occurrence of an event at time *t*, given survival up to that time. The cumulative hazard function is given by;

$$H(y) = \int_0^y \log(y) du = \log(S(y)) \tag{48}$$

In the subsequent subsections, the *H*(*y*) and *S*(*y*) for the proposed probability distribution in the will be derived.

S(y) and H(y) for IESStD

Substituting (5) into (47), the survival function of IESStD is obtained and expressed as follows;

$$S(y) = 1 - \left[\frac{1}{2} + \frac{y}{2\sqrt{\lambda + y^2}}\right]^{\frac{1}{u}} \tag{49}$$

To obtain the cumulative hazard function of IESStD, we make the substitution of equation (49) into equation (48) and in figure 3, the hazard function plot is presented.

$$H\left(y\right)=\log\left\{1-\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^{2}}}\right]^{\frac{1}{u}}\right\} \tag{50}$$

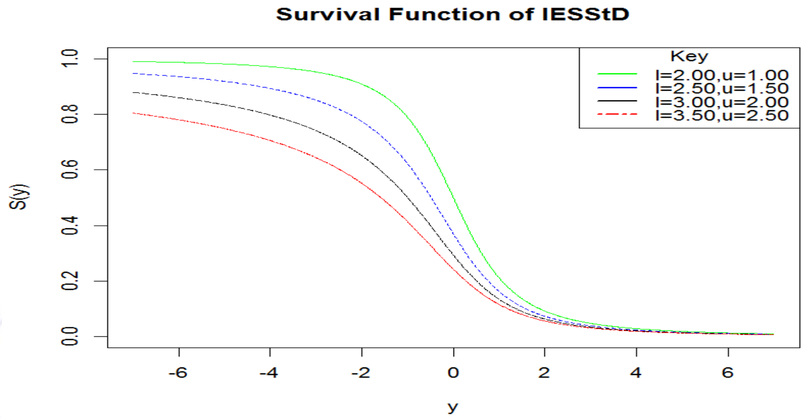

Figure 3: Survival Function plot of IESStD at Different Parameter values where $l=\lambda$

Hazzard Function

The hazard function is one important function in reliability analysis. This can be defined as a ratio of the probability density function to the survival function. Utilizing the densities of the proposed distributions, the hazard function would be derived.

$$H_{z}\left(y\right)=\frac{g\left(y\right)}{S\left(y\right)} \tag{51}$$

Hazzard function for IESStD

For a random variable with IESStD density, the hazard function can be obtained by substituting (6) for g(y) and equation (49) for S(y) into equation (51). The plot of the function is presented in figure 4. This is illustrated as follows;

$$H_{z(IE)}\left(y\right)=\frac{\frac{\lambda}{u}\left[\left(\frac{1}{2}+\frac{y}{\sqrt{\lambda+y^{2}}}\right)\right]^{\frac{1-u}{u}}\cdot\frac{1}{\left(\lambda+y^{2}\right)^{\frac{3}{2}}}}{1-\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^{2}}}\right]^{\frac{1}{u}}} \tag{52}$$

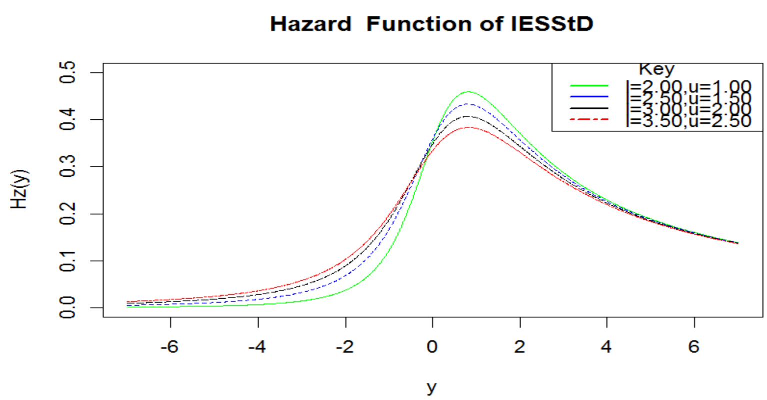

Figure 4: Hazard Function plot of IESStD at Different Parameter values where $l=\lambda$

Information Measures

Information measures are quantities used to quantify various aspects of information, uncertainty, and entropy in information theory. These measures have had different applications in various fields including communication theory, cryptography, machine learning, and statistics, where understanding and quantifying information and uncertainty are crucial (Pierce 2012). In this study the Renyi entropy and its Arimoto measure will be derived in this subsection.

Renyi entropy

In information theory, the Rényi entropy family of entropy measures bear the name of the mathematician Alfréd Rényi. It generalizes the concept of Shannon entropy, providing a parameterized family of entropy measures. This can be defined as follows;

$$I_R(c)=\frac{1}{1-c}\log\int_0^{\infty} g^c(y)dy \text{ ; for } c>1, c\neq 1 \tag{53}$$

Renyi entropy for IESStD

Suppose a random variable $y \sim IESStD(\lambda,u)$, then the degree of uncertainty can be estimated by substituting (6) for $g(y)$ in equation (53) as follows;

$$I_{R(IE)}(c)=\frac{1}{1-c}\log\int_0^{\infty}\left(\frac{1}{u}\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1-u}{u}}\frac{2\lambda}{4\left(\lambda+y^2\right)^{\frac{3}{2}}}\right)^c dy$$

$$I_{R(IE)}(c)=\frac{1}{1-c}\log\left(\left(\frac{\lambda}{2u}\right)^c\left(\frac{1}{2}\right)^{\frac{c(1-u)}{u}}\int_0^{\infty}\left[1+\frac{y}{\sqrt{\lambda+y^2}}\right]^{\frac{c(1-u)}{u}}\left(\lambda+y^2\right)^{-\frac{3c}{2}}dy\right) \tag{54}$$

Expanding $\left[1+\frac{y}{\sqrt{\lambda+y^2}}\right]^{\frac{c(1-u)}{u}}$ via the binomial expansion, the following is obtained;

$$\left[1+\frac{y}{\sqrt{\lambda+y^2}}\right]^{\frac{c(1-u)}{u}} = \sum_{q=0}^{\frac{c(1-u)}{u}} \begin{pmatrix} \frac{c(1-u)}{u} \\ q \end{pmatrix} (1)^{\frac{c(1-u)}{u}-q} y^q \left(\lambda+y^2\right)^{-\frac{q}{2}} \tag{55}$$

Substituting into (54), we have:

$$I_{R(IE)}(c) = \frac{1}{1-c}\log\left(\left(\frac{\lambda}{2u}\right)^c \left(\frac{1}{2}\right)^{\frac{c(1-u)}{u}} \int_0^\infty \sum_{q=0}^{\frac{c(1-u)}{u}} \begin{pmatrix} \frac{c(1-u)}{u} \\ q \end{pmatrix} (1)^{\frac{c(1-u)}{u}-q} y^q \left(\lambda+y^2\right)^{-\frac{q}{2}} \left(\lambda+y^2\right)^{-\frac{3c}{2}} dy \right) \tag{56}$$

$$I_{R(IE)}(c) = \frac{1}{1-c}\log\left(\left(\frac{\lambda}{2u}\right)^c \left(\frac{1}{2}\right)^{\frac{c(1-u)}{u}} \alpha_{(IE)} \int_0^\infty y^q \left(\lambda+y^2\right)^{-\frac{q-3c}{2}} dy \right) \tag{57}$$

Where $\alpha_{(IE)} = \sum_{q=0}^{\frac{c(1-u)}{u}} \begin{pmatrix} \frac{c(1-u)}{u} \\ q \end{pmatrix} (-1)^{\frac{c(1-u)}{u}-q}$

Since it is known that $\beta[a,b] = \int_0^\infty \frac{y^{a-1}}{(1+y)^{a+b}} dy$, then by transforming (57) into its $\beta[a,b]$ equivalence we have that;

$$I_{R(IE)}(c) = \frac{1}{1-c}\log\left(\left(\frac{\lambda}{2u}\right)^c \left(\frac{1}{2}\right)^{\frac{c(1-u)}{u}} \alpha_{(IE)} \int_0^\infty \frac{y^{(q+1)-1}}{\left(\lambda+y^2\right)^{\frac{q+3c-2}{2}+1}} dy \right)$$

$$I_{R(IE)}(c) = \frac{1}{1-c}\log\left(\left(\frac{\lambda}{2u}\right)^c \left(\frac{1}{2}\right)^{\frac{c(1-u)}{u}} \alpha_{(IE)} \beta\left[(q+1), \left(\frac{q+3c-2}{2}\right) \right] \right) \tag{58}$$

Arimoto measure of entropy

The Arimoto measure of entropy, also known as the "information radius entropy" or "information divergence entropy," provides a way to compare the uncertainty or disorder in probability distributions. This is defined by the following equation;

$$A_\varepsilon(c) = \frac{c}{1-c}\left[\left(E_c(\theta)\right)^{\frac{1}{c}} - 1 \right] = \frac{c}{1-c}\left(\int_0^\infty \left(g^c(y)dy\right)^{\frac{1}{c}} - 1 \right); \text{ for } c>1, c \neq 1 \tag{59}$$

Arimoto measure of entropy for IESStD

Suppose a random variable $y \sim IESStD(\lambda,u)$, then on substituting (18) into (59), the Arimoto measure of entropy is derived as follows;

$$A_{\varepsilon(IE)}(c)=\frac{c}{1-c}\left(\left(\int_0^\infty \left(h_{u,\lambda}\right)^c y^{ci}\left[\lambda+y^2\right]^{-c\left(\frac{3+i}{2}\right)}dy\right)^{\frac{1}{c}}-1\right) \tag{60}$$

$$A_{\varepsilon(IE)}(c)=\frac{c}{1-c}\left(\left(\left(h_{u,\lambda}\right)^c \int_0^\infty \frac{y^{ci}}{\left[\lambda+y^2\right]^{c\left(\frac{3+i}{2}\right)}}dy\right)^{\frac{1}{c}}-1\right)$$

$$A_{\varepsilon(IE)}(c)=\frac{c}{1-c}\left(\left(\left(h_{u,\lambda}\right)^c \frac{1}{\lambda}\int_0^\infty \frac{y^{ci}}{\left[1+\frac{y^2}{\lambda}\right]^{c\left(\frac{3+i}{2}\right)}}dy\right)^{\frac{1}{c}}-1\right) \tag{61}$$

By substituting the transformation in (27) into (61) we have that;

$$A_{\varepsilon(IE)}(c)=\frac{c}{1-c}\left(\left(\left(h_{u,\lambda}\right)^c \frac{1}{\lambda}\int_0^\infty \frac{(p\lambda)^{\frac{ci}{2}}}{\left[1+p\right]^{c\left(\frac{3+i}{2}\right)}}\frac{\lambda}{2(p\lambda)^{\frac{1}{2}}}dp\right)^{\frac{1}{c}}-1\right)$$

$$A_{\varepsilon(IE)}(c)=\frac{c}{1-c}\left(\left(\left(h_{u,\lambda}\right)^c \frac{1}{\lambda}\frac{\lambda^{\frac{ci-1+1}{2}}}{2}\int_0^\infty \frac{p^{\frac{ci-1}{2}}}{\left[1+p\right]^{c\left(\frac{3+i}{2}\right)}}dp\right)^{\frac{1}{c}}-1\right)$$

$$A_{\varepsilon(IE)}(c)=\frac{c}{1-c}\left(\left(\left(h_{u,\lambda}\right)^c \frac{\lambda^{\frac{ci-1+1}{2}-1}}{2}\int_0^\infty \frac{p^{\frac{ci-1}{2}}}{\left[1+p\right]^{c\left(\frac{3+i}{2}\right)}}dp\right)^{\frac{1}{c}}-1\right) \tag{62}$$

Therefore, on transforming (62) into the form in (29), the Arimoto measure of entropy of the IESStD is given as;

$$A_{\varepsilon(IE)}(c)=\frac{c}{1-c}\left(\left(\left(h_{u,\lambda}\right)^{c}\frac{\lambda^{\frac{ci-2}{2}}}{2}\beta\left[\frac{ci+1}{2},c\left(\frac{3+i-2}{2}\right)\right]\right)^{\frac{1}{c}}-1\right) \tag{63}$$

Parameter Estimation and Stability Analysis for the IESS*t*D

This section presents the MLE estimates and a stability analysis of the estimates of the proposed model. The method of maximum likelihood was used to estimate the values of the model's parameters. This will be illustrated in the following subsections.

Maximum Likelihood Estimators (MLE)

The Maximum likelihood estimation method (MLE) have been known to yield estimators with properties viz; minimum bias, efficiency, consistency etc. (Banerjee and Bhunia 2022). Let random variable $y_1,y_2,\ldots,y_n$ of random sample of size n be obtained from IESStD with probability density function in equation (6), then, the likelihood function is given by:

$$L(\theta)=\prod_{i=1}^{n}\left[\frac{\lambda}{2u\left(\lambda+y^2\right)^{\frac{3}{2}}}\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1-u}{u}}\right] \tag{64}$$

$$L(\theta)=\left(\frac{\lambda}{2u}\right)^{n}\sum_{t=1}^{n}\left(\lambda+y^2\right)^{-\frac{3}{2}}\sum_{n}^{n}\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right]^{\frac{1-u}{u}}$$

$$\log\left(L(\theta)\right)=n\log\left(\frac{\lambda}{2u}\right)-\frac{3}{2}\sum_{t=1}^{n}\log\left(\lambda+y^2\right)+\frac{1-u}{u}\sum_{n}^{n}\log\left[\frac{1}{2}+\frac{y}{2\sqrt{\lambda+y^2}}\right] \tag{65}$$

The Mean Squared Error (MSE) and Root Mean Square Error (RMSE) serve as critical metrics in statistics and data analysis. These metrics offer a comprehensive evaluation of the accuracy and precision of predictions generated by statistical models or estimators (Chai and Draxler 2014, Hodson 2022). For the proposed probability distributions, the stability of the Maximum Likelihood Estimators (MLE) was studied (with the aid of r software) by observing the behaviors of the MSE and RMSE at different sample sizes via Monte Carlo simulations. Random numbers were generated using the quantile functions and the uniform distribution with fixed parameter values for the simulations.

The outcomes of these simulations are summarized in Table 1, showcasing MSE and RMSE values for various parameter settings. These results provide a concise overview of how the model or estimator performs under different sample sizes.

3. Results

Stability Analysis Via Montecarlo Simulations for IESStD

Table 4.1: Stability measures for IESStD

N	Parameter Type	MLE	Absolute Bias	MSE	RMSE
		λ = 0.40, u = 0.01			
30	u	0.01162	0.00162	0.00036	0.01894

	λ	0.57381	0.17381	1.11758	1.05716
100	u	0.01046	0.00046	0.00017	0.01316
	λ	0.46523	0.06523	0.39845	0.63123
700	u	0.00949	0.00051	0.00005	0.00675
	λ	0.38878	0.01122	0.08326	0.28855
900	u	0.00957	0.00043	0.00004	0.00622
	λ	0.39033	0.00967	0.07022	0.26498
1000	u	0.00962	0.00038	0.00004	0.00607
	λ	0.39237	0.00763	0.06687	0.25859
2000	u	0.00960	0.00040	0.00002	0.00442
	λ	0.38779	0.01221	0.03451	0.18577
3000	u	0.00967	0.00033	0.00001	0.00364
	λ	0.38980	0.01020	0.02332	0.15271
5000	u	0.00979	0.00021	0.00001	0.00286
	λ	0.39330	0.00670	0.01431	0.11962

The findings from the simulations, revealed that as the sample size was systematically increased, a consistent downward trajectory was observed in the MSE and RMSE values. The results also revealed that the MLE estimates approaches the true parameter values with increase in sample size. This noteworthy trend implies that with larger sample sizes, the stability of the estimations for the parameters of the IESStD substantially improved. This behavior aligns with a fundamental principle of the central limit theorem (CLT), where larger sample sizes tend to yield more precise estimates. The decreasing trend in MSE and RMSE reflects a higher degree stability of model estimators (Chai and Draxler, 2014; Hodson, 2022).

Application

The first dataset consists of daily ASI_NGN prices collected from March 5, 2023, to September 29, 2023, with a total of 103 observations corresponding to 103 trading days. The second dataset pertains to Ethereum daily prices obtained over the period from November 21, 2023, to December 21, 2023, comprising 31 observations, each corresponding to a trading day. The third dataset encompasses Dow Jones Industrial Average daily prices collected from August 1, 2023, to December 20, 2023, with 100 observations, each corresponding to a trading day. Finally, the fourth dataset concerns SandP500 daily prices gathered from August 28, 2023, to December 20, 2023, comprising 81 observations, each corresponding to a trading day. Subsequently, returns were computed for each dataset. The returns were calculated using the formula;

$$\gamma_t = \log\left(\frac{\rho_t}{\rho_{t-1}}\right) \tag{66}$$

where ρ_t and ρ_{t-1} be the recent and immediate past price index observed and (γ_t) is the compounded return series. Tables 1, 3, 5, and 7 provide a comprehensive summary of key statistical metrics, including the mean, standard error, standard deviation, kurtosis, skewness, minimum, and maximum values for the respective datasets.

ASI_NGN data

Table 1: Descriptive statistics for ASI_NGN

	Price	Returns
Mean	61673.67	0.001005
Standard Error	539.5818	0.000452

Standard Deviation	5476.157	0.00459
Kurtosis	-1.0748	5.411266
Skewness	-0.62128	1.288912
Minimum	52109.43	-0.01086
Maximum	68359.22	0.02215
Count	103	102

-0.000434782, -0.001328262, -0.001499114, -0.00286031, -0.006063438, -0.00041062, -0.000149324, 0.003073372, 0.003091281, 0.000309484, -0.000429711, 0.004235045, -0.003472784, -0.005433004, 0.000390409, -0.001300448, -0.000307711, 0.000353123, 0.004809365, 0.006337209, 0.000714917, -0.000332002, 0.002219645, 0.003907184, 0.00104189, -0.00060452, 2.82e-05, 0.001902522, 0.003064389, 0.000538171, 0.00025865, -0.002036143, -0.001878883, -0.000764415, 0.000798648, 0.006968856, -0.007662982, -0.000180504, 0.000923054, -0.000432626, 0.006676973, 0.0005082, -0.000982056, -0.004825644, -0.00283801, -0.001352515, -0.002004356, 0.004782673, 0.001766162, 0.007226411, 0.001180155, -6.46e-05, 0.005644225, 0.002585572, -0.001242113, -0.008891923, -0.010864004, 0.007104996, 0.010635126, 0.007091276, 0.003489393, 0.005745243, -0.008737945, 0.006930358, 0.006161407, 0.005603999, 0.000968125, -0.001135256, 0.000275538, 0.001568954, 0.000690453, 0.000111575, -0.001427488, -0.005756822, 0.013392481, 0.016998554, -0.000198889, -0.000534966, -0.000103008, 0.001799837, -0.000122395, 0.000113368, 0.000303367, 0.000183353, 0.022149901, 0.001249902, -0.00087032, 0.002521536, 0.002085306, 0.001505288, 0.000653749, -0.003911362, 0.001336219, 0.001560714, 0.00013863, 0.000444215, -0.000400047, -0.003291537, 0.000222874, 0.000934358, 0.001457388, 0.000689728, -0.000760235

Table 2: Goodness of fit for ASI_NGN Data

	MLE	**AIC**	**Log Likelihood**	**Rank**
IESStD	0.0002815 0.1475543	-1483.494	743.7469	1
ESStD	0.1023087 0.0666794	-1478.8887	741.4443	2
GD	0.0000002 0.004721	-808.1737	406.0868	5
Logistic D	0.000698 0.0021952	-834.7997	419.3999	4
SStD	8.76E-06	-851.3689	426.6845	3

Ethereum Data

Table 3: Descriptive statistics for Ethereum Data

	Price	Returns
Mean	2177.8877	0.0019
Standard Error	20.2592	0.0020
Standard Deviation	112.7985	0.0108
Kurtosis	-0.7772	1.1123
Skewness	-0.1411	0.0479
Minimum	1934.6200	-0.0241
Maximum	2358.4200	0.0279
Count	31	30

-0.000372887, 0.004908589, -0.00817196, 0.004280193, -0.006285743, 0.001690532, -0.018237868, 0.010528015, 0.011031944, -0.00428529, -0.024088949, 0.0021674, -0.003303226, 0.000482732, 0.023162183, -0.011492212, 0.009534629, 0.009936775, 0.005581269, 0.015849626, 0.007481924, 0.004804316, -0.004149701, 0.004466603, -0.007460431, -0.004284598, 0.000515265, 0.003920958, -0.000237942, 0.02791408

Table 4: Goodness of fit for Ethereum Data

	MLE	**AIC**	**Log Likelihood**	**Rank**
IESStD	0.04498 0.02331	-1484.88	744.4401	1
ESStD	0.0000005 0.4471338	-843.7376	423.8688	3
GD	1.733 0.5156	-1476.441	740.2206	2
Logistic D	0.0018296 0.005711	-185.1427	94.5713	4
SStD	0.0000953	-184.7956	93.3978	5

Dow Jones Industrial Average data

Table 5: Descriptive statistics for Dow Jones Industrial Average

	Price	Returns
Mean	34640.3959	0.0002
Standard Error	112.4059	0.0003
Standard Deviation	1124.0593	0.0029
Kurtosis	0.0796	-0.1201
Skewness	0.5584	0.1638
Minimum	32418.0500	-0.0056
Maximum	37557.9800	0.0073
Count	100	99

-0.005534509, 0.002924123, -3.80658e-05, 0.000709946, 0.001839671, 0.006047309, 0.002060594, 0.001877715, 0.001562411, 0.000759889, -0.000843932, -0.000959268, -0.000492261, 0.003543603, 0.006332598, 0.000167226, 0.001025238, -0.000702127, 0.001447991, 0.002277363, -0.00077511, 0.002523934, 2.08781e-05, -0.000566459, 0.002034166, 0.006153972, 0.000690737, 0.004986988, -0.002824271, -0.000508947, 0.000724666, 0.000440175, 0.002842542, 0.00730635, 0.002905115, 0.001631167, 0.006789412, -0.004879027, -0.003320622, -0.001383265, 0.002694465, -0.002509524, -0.003736459, -0.003261814, -0.00426998, 0.000175935, 0.004029419, 0.000506171, -0.00223691, 0.000842431, 0.001736691, 0.002554359, 0.003760327, -0.000127439, 0.001658183, -0.005625949, -0.000965501, -0.002047929, 0.001492128, -0.000879589, -0.00498945, 0.000545779, -0.001352137, -0.004697645, -0.000974656, -0.00133173, 6.89923e-05, -0.003607238, 0.00414789, -0.00090017, -0.000217172, 0.001097496, 0.00095927, 0.000713185, -0.002485839, -0.002449128, 0.001459113, -0.002116328, 0.000472008, 0.00366431, 0.002685932, 0.003135072, -0.004740586, 0.002338619, -0.002215649, -0.000471912, 0.000352343, -0.003659811, -0.002257931, -0.004460407, 0.000322757, 0.001297505, 0.000652249, -0.002356884, -0.001946572, 0.005017988, -0.001857147, -0.000820928, -0.004264514

Table 6: Goodness of fit for Dow Jones Industrial Average

	MLE	**AIC**	**Log Likelihood**	**Rank**

IESStD	1.089515 0.07378	-1484.8801	744.4401	1
ESStD	0.000716 1.700836	-1482.6829	743.3415	2
GD	1.534059 0.660243	-1475.0845	739.5422	3
Logistic D	0.000151 0.00162	-874.0354	439.0177	4
SStD	0.000008	-864.2681	433.134	5

SandP500 data

Table 7: Descriptive statistics for SandP500

	Price	Returns
Mean	4429.8331	0.0003
Standard Error	16.7623	0.0004
Standard Deviation	150.8608	0.0034
Kurtosis	-0.5507	-0.0342
Skewness	0.1282	-0.1309
Minimum	4117.3700	-0.0072
Maximum	4768.3700	0.0082
Count	81	80

-0.006162143, 0.002540299, 0.001962185, -3.31286e-05, 0.001148089, 0.005888347, 0.001992872, 0.001701067, 0.001774994, 0.003446826, -0.001699733, -0.000247164, -0.002355251, 0.002545365, 0.001638425, -0.00041114, 0.000425455, -0.000849542, 0.000259168, 0.001760138, -0.000878573, 0.003197768, 0.000556451, 0.000516655, 0.00069305, 0.008206117, -0.00036311, 0.006729709, -0.003525065, 0.00043622, 0.001231709, 0.000760635, 0.004060597, 0.008113905, 0.00453889, 0.002802975, 0.005184844, -0.002089769, -0.005169438, -0.006272688, 0.003144071, -0.000732639, -0.00550042, -0.003699759, -0.0058588, -4.27004e-05, 0.004576864, -0.002185143, -0.002721267, 0.001860443, 0.002255925, 0.00272912, 0.005101062, -0.000566697, 0.003507837, -0.006010379, 3.44339e-05, -0.001178314, 0.002551817, 9.95804e-05, -0.006446723, 0.001743702, -0.000998118, -0.007181873, -0.004099394, -0.000935218, 0.000313142, -0.005313195, 0.003645707, 0.000538896, -0.002480758, 0.002910213, 0.000619123, -0.001396894, -0.003038321, -0.001825344, 0.000780663, -0.000694099, 0.001661525, 0.006255625

Table 8: Goodness of fit for SandP500

	MLE	**AIC**	**Log Likelihood**	**Rank**
IESStD	0.0331508 0.052161	-1483.4938	743.7469247	1
ESStD	0.0011795 159.47111	-1480.7213	742.3606304	2
GD	1.496831 0.5965827	-1477.8274	740.9137114	3
Logistic D	0.0004008 0.0019087	-678.33374	341.1668708	4
SStD	1.13E-05	-670.9329	336.4664	5

Table 2 presents the results of fitting the ASI_NGN return series to five probability models (IESStD, ESStD, GD, Logistic D, and SStD). The AIC, MLE, and log-likelihood values for each model are provided, revealing that the IESStD model exhibits the smallest AIC and the highest log-likelihood values. A larger likelihood value indicates a better model fit, while a lower AIC suggests better fit. Therefore, based on the presented results, the IESStD model emerges as the most suitable among the other probability models for ASI_NGN return series.

Table 4, 6, and 8 presents the results of fitting the Ethereum, Dow Jones Industrial Average, SandP500 return series to IESStD, ESStD, GD, Logistic D, and SStD probability models. The results include the AIC, MLE, and log-likelihood values. The results revealed that the IESStD has the smallest AIC and a corresponding high log-likelihood values. A larger likelihood value indicates a better model fit, while a lower AIC which also suggests better fit indicating that the IESStD model provides a better fit when compared with the other probability models.

4. Conclusion

IESSt distribution was derived in this study by inverting the CDF generator of the exponentiated family of distribution. Probability density plot and the corresponding cumulative distribution function were plotted for some selected parameter values. Some statistical properties of the model such as the moments, the moment generating function, the quantile function, the c^{th} order statistic were derived for the proposed distribution. The model parameters were estimated using the maximum likelihood estimation (MLE) method. Through the use of AdequacyMlodel and Maxlik packages in r software, a Montecarlo simulation of the the MLE was performed and the results revealed the MLE to be consistent and as expected, with increased sample size, the RMSE almost converges to zero. The model potential of the in financial data analysis was illustrated using four financial data (return of daily price of Ethereum, ASI, SandP500 and Dow Jones Industrial Average) datasets. The results revealed that the IESSt distribution fitted well among other distributions in terms of having the lowest AIC and the largest loglikelihood values. For modeling financial data, the application of IESSt distribution should be highly considered. Also, in future research, this distribution may contribute in the area of financial asset modeling such as the Generalized Autoregressive Conditional Heteroscedasticity (GARCH), in reliability and survival analysis especially in observations that are right skewed.

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