



Fourier Transformation and Hyers-Ulam Stability of Linear Differential Equations in $L^2(\mathbb{R})$

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ABSTRACT

This paper investigates the Hyers-Ulam stability of linear differential equations with constant coefficients in $L^2(\mathbb{R})$ using Fourier transform methods. By converting differential operators into algebraic multipliers in the frequency domain, we obtain sufficient conditions for the existence of solutions and for stability under perturbations. Plancherel's theorem provides the key norm estimate linking the frequency and time domains. Under a bounded inverse multiplier condition, we prove Hyers-Ulam stability for both second-order and general n th-order equations. Examples involving a Gaussian input, an RLC circuit, and a resonant oscillator illustrate the scope of the theory and show where stability fails.

1 Introduction

Stability is a fundamental concept in the qualitative theory of differential equations because mathematical models are frequently subject to perturbations arising from measurement errors, approximation procedures, numerical methods, and uncertainties in model parameters. Therefore, it is natural to ask whether an approximate solution remains close to an exact solution. This question is central to Hyers-Ulam stability.

Hyers-Ulam stability asks whether approximate solutions of a functional equation remain close to exact solutions. The idea began with Ulam's question on additive mappings and was answered by Hyers in Banach spaces (Hyers, 1941; Young, 1958; Rus, 2009b). Since then, the notion has been extended to differential equations and other analytic structures (Jung, 2011; Găvruta et al., 2011). Ante et al. (2025) established sufficient conditions for generalized Ulam-Hyers stability for Caputo fractional derivatives with nonlocal conditions

For a differential operator D , if an approximate solution y satisfies $\|Dy - f\| \leq \varepsilon$, $\varepsilon > 0$, Hyers-Ulam stability requires an exact solution y_a satisfying $Dy_a = f$ and $\|y - y_a\| \leq M\varepsilon$, where $M > 0$ is called a Hyers - Ulam stability constant (Rezaie et al. 2021).

Fourier analysis is especially effective for linear differential equations with constant coefficients because derivatives become multipliers in the frequency domain (Bracewell, 1999; Katznelson, 2002). In $L^2(\mathbb{R})$, this approach is particularly natural since Plancherel's theorem allows one to transfer norm estimates directly between time and frequency variables (Rudin, 1987; Reed & Simon, 1980). Recent work has applied Fourier methods to Hyers-Ulam

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stability of constant-coefficient, nonlinear, higher-order, and related boundary-value problems. Ramdoss et al. (2020) treated n th-order constant-coefficient equations; Mohanapriya et al. (2020) considered Mittag–Leffler–Hyers–Ulam stability; and Rezaie et al. (2021) studied an $L^1(\mathbb{R})$ setting.

The use of Fourier methods has subsequently been extended beyond ordinary differential equations. Kang and Kim (2022) developed an L^2 – based Fourier-transform method for generalized Hyers–Ulam stability of the diffusion equation. Their work is particularly relevant to the present study because it demonstrates that Fourier analysis can provide stability estimates naturally formulated in the L^2 – norm rather than only in the supremum norm.

Further developments have considered more general classes of equations. Ciplea et al. (2023) investigated Hyers–Ulam stability for a general linear partial differential equation in a Banach-space setting, illustrating the continuing extension of stability theory beyond ordinary differential equations. Selvam et al. (2024) applied Fourier transforms to first- and second-order nonlinear differential equations and established Hyers–Ulam, Hyers–Ulam–Rassias, Mittag–Leffler–Hyers–Ulam, and Mittag–Leffler–Hyers–Ulam–Rassias stability results. Their study demonstrates that Fourier-transform techniques can be extended beyond the constant-coefficient linear framework to nonlinear and generalized stability settings.

More recently, Sivashankar et al. (2025) investigated the stability of higher-order linear differential equations using the Fourier transform, including Hyers–Ulam and Hyers–Ulam–Mittag–Leffler stability. Their work also considered applications to mass–spring systems, thereby demonstrating the applicability of Fourier-based stability analysis to mathematical models arising in mechanics. Kim et al. (2025) further developed a generalized Hyers–Ulam stability framework for the Laplace equation with Dirichlet boundary conditions on a half-space. Their Fourier-based approach simultaneously considered equation and boundary errors and produced explicit stability estimates, illustrating the continuing development of transform-based stability methods for higher-dimensional boundary-value problems.

Despite these developments, an important issue remains in the Fourier-transform treatment of Hyers–Ulam stability. The aim of this paper is to establish existence and Hyers–Ulam stability results for such equations in the square-integrable setting.

2. Fourier Transform and Plancherel’s Theorem

For $f \in L^1(\mathbb{R})$, define the Fourier transform by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt. \quad (1)$$

The inverse Fourier transform is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} d\omega. \quad (2)$$

These conventions are standard in harmonic analysis (Bracewell, 1999; Deitmar, 2005).

Consider the n th – order linear differential equation

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) = f(t), t \in \mathbb{R}. \quad (3)$$

Its characteristic polynomial is

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k. \quad (4)$$

The equation (3) is Hyers–Ulam stable in $L^2(\mathbb{R})$ if every approximate solution with small residual is close to an exact solution in the L^2 -norm (Rezaie et al., 2021).

Theorem 1: Plancherel's theorem

The Fourier transform extends uniquely to $L^2(\mathbb{R})$ as an isometry up to the chosen normalization. In particular,

$$\|\hat{f}\|_2 = \sqrt{2\pi} \|f\|_2, \quad (5)$$

and

$$\|f\|_2^2 = \frac{1}{2\pi} \|\hat{f}\|_2^2. \quad (6)$$

Thus, the Fourier transform preserves the L^2 -structure up to a constant factor (Stein & Shakarchi, 2003).

Remark

Plancherel's theorem is the central estimate in the paper. It allows the error in the time domain to be controlled by the error in the frequency domain, which is exactly what makes the Hyers–Ulam argument work.

3. Hyers–Ulam Stability of the Linear Differential Equation

Theorem 2: Second-order Hyers–Ulam stability

Consider the second-order linear differential equation

$$y''(t) + a_1 y'(t) + a_0 y(t) = f(t), f \in L^2(\mathbb{R}). \quad (7)$$

Assume

$$p(z) = z^2 + a_1 z + a_0 \quad (8)$$

satisfies

$$M = \sup_{\omega \in \mathbb{R}} \frac{1}{|p(i\omega)|} < \infty. \quad (9)$$

If $y \in L^2(\mathbb{R})$ satisfies

$$\|y'' + a_1 y' + a_0 y - f\|_2 \leq \varepsilon, \quad \varepsilon > 0 \quad (10)$$

then there exists an exact solution $y_a \in L^2(\mathbb{R})$ such that

$$\|y - y_a\|_2 \leq M\varepsilon. \quad \text{where } M, \varepsilon > 0 \quad (11)$$

Hence, the equation is Hyers–Ulam stable in $L^2(\mathbb{R})$ (Li & Shen, 2010; Impean & Popa, 2010).

Proof

Let the differential operator D be define by

$$Dy = y'' + a_1y' + a_0y \quad (12)$$

Suppose

$$Dy = 0 \quad (13)$$

That is,

$$y'' + a_1y' + a_0y = 0 \quad (14)$$

Taking Fourier transform gives

$$p(i\omega)\hat{y}(\omega) = 0 \quad (15)$$

Since the characteristics polynomial is

$$p(z) = z^2 + a_1z + a_0$$

Under the assumption,

$$\inf_{\omega \in \mathbb{R}} |p(i\omega)| > 0 \quad (16)$$

we have $p(i\omega) \neq 0$ for every $\omega \in \mathbb{R}$. Hence $\hat{y}(\omega) = 0$, and by the uniqueness of the Fourier transform, $y = 0$ in $L^2(\mathbb{R})$. Thus, the homogeneous equation $Dy = 0$ has only the trivial solution in $L^2(\mathbb{R})$ (Stein & Shakarchi, 2003).

Define

$$h(t) = y''(t) + a_1y'(t) + a_0y(t) - f(t). \quad (17)$$

Then $\|h\|_2 \leq \varepsilon$. Taking Fourier transforms gives

$$\hat{h}(\omega) = p(i\omega)\hat{y}(\omega) - \hat{f}(\omega). \quad (18)$$

Set

$$\widehat{y_a}(\omega) = \frac{\hat{f}(\omega)}{p(i\omega)}. \quad (19)$$

Then

$$\hat{y}(\omega) - \widehat{y_a}(\omega) = \frac{\hat{h}(\omega)}{p(i\omega)}. \quad (20)$$

By Plancherel's theorem,

$$\|y - y_a\|_2 = \left\| \frac{\hat{h}}{p(i\omega)} \right\|_2 \leq \left\| \frac{1}{p(i\omega)} \right\|_\infty \|\hat{h}\|_2. \quad (21)$$

Using the Fourier isometry, we obtain

$$\|y - y_a\|_2 \leq M\varepsilon. \quad \text{where } M, \varepsilon > 0$$

This completes the proof (Ramdoss et al., 2020; Rezaie et al., 2021).

Theorem 3: n th-order Hyers–Ulam stability

Consider the n th – order linear differential equation

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) = f(t), f \in L^2(\mathbb{R}). \quad (22)$$

Assume that

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k \quad (23)$$

satisfies

$$M = \sup_{\omega \in \mathbb{R}} \frac{1}{|p(i\omega)|} < \infty. \quad (24)$$

If $y \in L^2(\mathbb{R})$ satisfies

$$\left\| y^{(n)} + \sum_{k=0}^{n-1} a_k y^{(k)} - f \right\|_2 \leq \varepsilon, \quad (24)$$

then there exists an exact solution $y_a \in L^2(\mathbb{R})$ such that

$$\|y - y_a\|_2 \leq M\varepsilon. \quad \text{where } M, \varepsilon > 0$$

Therefore, the n th-order equation is Hyers–Ulam stable in $L^2(\mathbb{R})$ (Impean & Popa, 2010; Qarawani, 2013; Ramdoss et al., 2020).

Proof

Let

$$h(t) = y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) - f(t). \quad (25)$$

Then $\|h\|_2 \leq \varepsilon$. Fourier transformation gives

$$\hat{h}(\omega) = p(i\omega)\hat{y}(\omega) - \hat{f}(\omega). \quad (26)$$

Define

$$\widehat{y}_a(\omega) = \frac{\hat{f}(\omega)}{p(i\omega)}. \quad (27)$$

Hence,

$$\hat{y}(\omega) - \widehat{y}_a(\omega) = \frac{\hat{h}(\omega)}{p(i\omega)}. \quad (28)$$

Again, Plancherel's theorem implies

$$\|y - y_a\|_2 \leq M\|h\|_2 \leq M\varepsilon. \quad (29)$$

This proves the theorem.

4. Existence and Smoothness

Under the hypotheses of Theorem 3, the exact solution defined by

$$\widehat{y}_a(\omega) = \frac{\hat{f}(\omega)}{p(i\omega)} \quad (30)$$

belongs to $L^2(\mathbb{R})$. Since $1/p(i\omega)$ is bounded, the inverse Fourier transform defines $y_a \in L^2(\mathbb{R})$. The regularity of the multiplier implies that the solution is smooth away from the origin, so $y_a \in C^n(\mathbb{R} \setminus \{0\})$ (Deitmar, 2005).

5. Examples

Example 5.1: Gaussian forcing

Consider

$$y''(t) + 2y'(t) + 5y(t) = e^{-t^2}. \quad (31)$$

The characteristic polynomial is

$$p(z) = z^2 + 2z + 5, \quad (32)$$

whose roots are

$$z = -1 \pm 2i. \quad (33)$$

Since $p(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$, the multiplier $1/p(i\omega)$ is bounded. Hence a solution exists in $L^2(\mathbb{R})$ and the equation is Hyers–Ulam stable (Li & Shen, 2010; Rezaie et al., 2021).

Example 5.2: RLC circuit

For a series RLC circuit with $L = 1$, $R = 2$, and $C = 1$,

$$i''(t) + 2i'(t) + i(t) = v(t). \quad (34)$$

If $v(t) = e^{-t^2} \in L^2(\mathbb{R})$, then the same Fourier multiplier argument shows that the response is Hyers–Ulam stable in $L^2(\mathbb{R})$. Small input perturbations cause only proportionally small changes in the current.

Example 5.3: Resonance failure

Consider

$$y''(t) + y(t) = f(t). \quad (35)$$

Then

$$p(z) = z^2 + 1, p(i\omega) = 1 - \omega^2. \quad (36)$$

Because $p(i\omega) = 0$ at $\omega = \pm 1$, the inverse multiplier is unbounded. In this resonance case, the L^2 -stability argument fails, and solvability may fail as well (Impean & Popa, 2010; Ohira & Ohira, 2025).

6. Discussion

The results show that the Fourier transform converts the differential operator into a multiplication operator, so the problem reduces to whether $1/p(i\omega)$ stays bounded on the real line. If it does, Plancherel's theorem transfers the frequency-domain bound back to the time domain and yields Hyers–Ulam stability. If it does not, resonance prevents bounded inversion and stability fails. For the homogeneous equation, $Dy = 0$, taking the Fourier transform gives equation (15). Since have $p(i\omega) \neq 0$ for every $\omega \in \mathbb{R}$, it follows that $\hat{y}(\omega) = 0$, and hence $y = 0$ in $L^2(\mathbb{R})$. Thus, the associated homogeneous problem possesses only the trivial solution. This observation establishes the triviality of the kernel of the differential operator in the considered $L^2(\mathbb{R})$ setting and supports the uniqueness of the exact solution used in the Hyers–Ulam stability argument.

The use of $L^2(\mathbb{R})$ is natural for finite-energy systems and makes the theory relevant for applied mathematics and engineering (Reed & Simon, 1980; Oppenheim & Schaffer, 2010). The examples confirm both the strength of the method and its limitations.

7. Conclusion

This paper presents a Fourier – based approach to Hyers–Ulam stability for linear constant – coefficient differential equations in $L^2(\mathbb{R})$. Plancherel’s theorem provides the key norm identity needed for stability estimates. Under suitable bounded-inverse conditions, both second-order and n th-order equations are shown to be Hyers–Ulam stable. Resonance cases demonstrate where the method breaks down.

The stability constant M provides an explicit quantitative measure of the effect of perturbations on the solution. Consequently, the result does not merely establish qualitative Hyers–Ulam stability but also specifies an explicit bound on the distance between an approximate solution and a corresponding exact solution.

The results also provide a unified Fourier-domain interpretation of Hyers–Ulam stability in $L^2(\mathbb{R})$, linking the spectral properties of the characteristic polynomial, the triviality of the homogeneous kernel, the boundedness of the inverse multiplier, and the resulting quantitative stability estimate.

Future investigations may consider the extension of the present approach to variable-coefficient and nonlinear differential equations, as well as to fractional and nonlocal differential operators. Another important direction is the treatment of resonant cases in which $p(i\omega)$ possesses zeros on the real axis. In such cases, alternative multiplier estimates or suitable restrictions on the function space may be required. It would also be of interest to investigate whether analogous spectral criteria can be developed for stability in other function spaces, including $L^p(\mathbb{R})$ and Sobolev spaces.

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