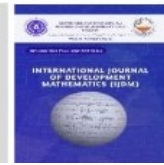




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A Modified Restricted Three-Body Model with a Rotating Mass Dipole and Triaxial Secondary

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Abstract

This study presents a modified restricted three-body model that incorporates a rotating mass dipole primary and a triaxial secondary to provide a more general description of motion in irregular binary celestial systems. The classical restricted three-body framework is extended by introducing the dipole half-separation, force-ratio parameter, and two triaxiality parameters associated with the non-spherical secondary. The governing equations of motion are formulated in a uniformly rotating coordinate system, and the corresponding effective potential is derived. From the proposed formulation, the mean motion is obtained, providing the fundamental dynamical relations required for studying the motion of the infinitesimal body. The formulation also accounts for the combined influence of the asymmetric mass distribution of the dipole primary and the geometric deformation of the secondary. Appropriate limiting cases are considered to demonstrate the consistency of the model with established restricted three-body formulations when the additional

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perturbing parameters vanish. The dynamics of test particles in binary systems resulting framework provides a unified involving irregular and non-spherical mathematical model for investigating the primary bodies.

1 Introduction

The restricted three-body problem (R3BP) is one of the fundamental models in celestial mechanics for describing the motion of a body of negligible mass under the gravitational influence of two finite primary bodies. In the classical formulation, the two primaries move under their mutual gravitational attraction, while the infinitesimal third body does not influence their motion. [Szebehely & Grebenikov \(1969\)](#) provided important foundations for the classical restricted three-body problem, while [Murray & Dermott \(1999\)](#) discussed its application to planetary and satellite dynamics. When the two primaries move in circular orbits about their common center of mass, the problem is referred to as the circular restricted three-body problem (CR3BP), and a uniformly rotating coordinate system provides a convenient description of the motion of the infinitesimal body relative to the primaries.

The classical CR3BP assumes that the primary bodies are spherical and possess uniform mass distributions. Although these assumptions simplify the mathematical formulation, they do not adequately represent many natural celestial bodies. Asteroids, cometary nuclei, and small planetary satellites frequently possess irregular shapes, elongated geometries, and non-uniform internal mass distributions. Such physical characteristics modify the gravitational field and may significantly influence the motion of nearby particles. [Yang et al. \(2018\)](#) and [Zeng et al. \(2016b\)](#), among others, showed that irregular geometry and asymmetric mass distribution can produce dynamical behavior that differs from that predicted by the classical point-mass approximation.

One important extension of the classical model is the inclusion of triaxiality, which accounts for the departure of a celestial body from spherical symmetry along its three principal axes. [Sharma et al. \(2001\)](#) investigated the influence of triaxiality in the restricted three-body problem and showed that the non-spherical geometry of the primaries affects the locations and stability properties of the equilibrium points. [Ansari et al. \(2018\)](#) considered heterogeneous spheroidal primaries, while [Gyegwe & Vincent \(2025\)](#); [Leke et al. \(2025\)](#); [Radwan & Abd El-Motelp \(2021\)](#) examined a triaxial elliptic restricted three-body problem. [Amuda & Oni \(2024\)](#); [Gyegwe et al. \(2022\)](#); [Kalantonis et al. \(2021\)](#) also studied triaxial primaries together with additional perturbative effects. These studies demonstrate that the geometrical structure of a primary body is an important factor in realistic celestial-mechanics models.

A different modification is provided by the rotating mass-dipole model, which offers a simple representation of elongated bodies and asymmetric mass distributions. [Zeng et al. \(2015\)](#) investigated the relationship between rotating mass-dipole models and naturally elongated celestial bodies, while [Zeng et al. \(2016a\)](#) extended the model by considering the oblateness of a primary. [Zeng & Liu \(2017\)](#) studied periodic orbits around irregular

asteroids using a dipole-based gravitational model, and [Leke et al. \(2024\)](#); [Zhang et al. \(2018\)](#) examined periodic orbits around a dipole-segment representation of dumbbell-shaped asteroids. [Barbosa Torres dos Santos et al. \(2017\)](#) and [Idrisi et al. \(2021\)](#) further demonstrated the use of mass-dipole representations in modified restricted three-body problems. More recently, [Abozaid et al. \(2024\)](#); [Abouelmagd \(2013\)](#); [Agbogidi et al. \(2025\)](#) considered a model involving a dipole primary together with oblateness, further illustrating the importance of dipole-type representations in the dynamics of irregular celestial systems.

Although rotating mass-dipole and triaxial effects have been studied in different forms of the restricted three-body problem, few studies have considered a system where the larger primary is a rotating mass dipole and the smaller primary is a triaxial body at the same time. This is the gap addressed in this study. Therefore, this study develops a modified restricted three-body model that combines these two effects in a single system. The model includes the mass parameter, dipole half-separation, force-ratio parameter, and triaxiality parameters. The study derives the kinetic and potential energies, mean motion, Hamiltonian, effective potential, and equations of motion. The model provides a basis for studying the motion of particles in binary systems with irregular and non-spherical bodies.

2 Mathematical Formulation

The present study considers the planar motion of an infinitesimal body of negligible mass m under the gravitational influence of two finite primaries, denoted by m_1 and m_2 . The larger primary m_1 is represented by a rotating mass dipole, while the smaller primary m_2 is assumed to possess a homogeneous triaxial ellipsoidal structure. The infinitesimal body does not influence the motion of the primaries, in accordance with the basic assumption of the restricted three-body problem. The primaries are assumed to revolve in circular orbits about their common center of mass with constant angular velocity n .

The formulation follows the classical rotating-frame description of the restricted three-body problem developed by [Szebehely & Grebenikov \(1969\)](#) and [Murray & Dermott \(1999\)](#), while the dipole representation and triaxial modifications are introduced to account for the asymmetric mass distribution and non-spherical geometry of the primaries.

2.1 Mass Distribution and Normalization

The larger primary is represented by two equal point masses m_{11} and m_{12} separated by a fixed distance $2L$. Therefore,

$$m_1 = m_{11} + m_{12}, \quad m_{11} = m_{12} = \frac{m_1}{2}. \quad (1)$$

Following [Szebehely & Grebenikov \(1969\)](#), the classical mass parameter of the restricted three-body problem is defined by

$$\mu = \frac{m_2}{m_1 + m_2}, \quad 0 < \mu \leq \frac{1}{2}, \quad (2)$$

where m_1 and m_2 denote the masses of the larger and smaller primaries, respectively.

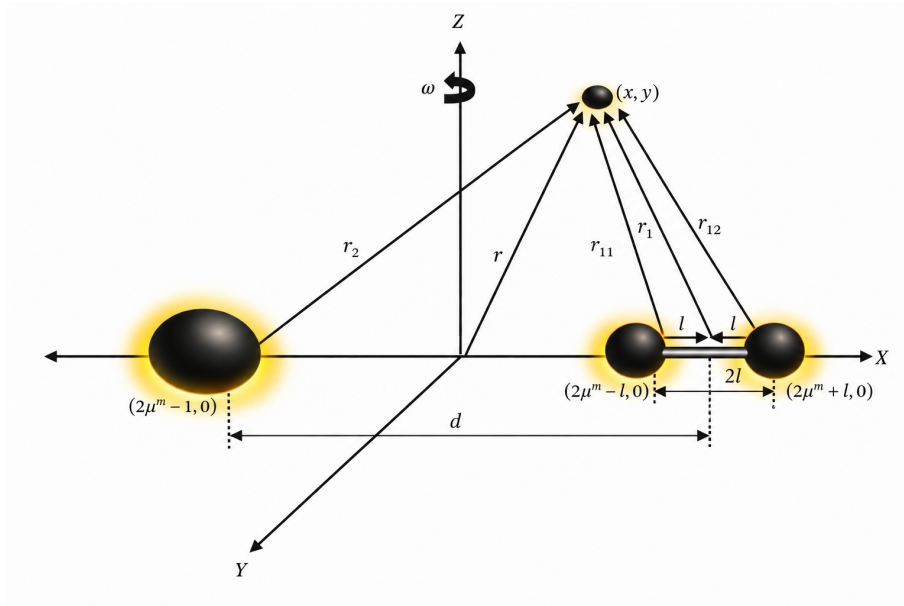


Figure 1. Model diagram of the modified restricted three-body problem

For convenience in the dipole formulation, and following the scaled dipole parametrization employed by [Abozaid et al. \(2024\)](#), a scaled mass parameter μ^m is introduced as

$$\mu^m = \frac{m_2}{2(m_{11} + m_{12} + m_2)}. \quad (3)$$

Since

$$m_1 = m_{11} + m_{12}, \quad (4)$$

equation (3) becomes

$$\mu^m = \frac{m_2}{2(m_1 + m_2)}. \quad (5)$$

Using equation (2),

$$\mu^m = \frac{\mu}{2}, \quad (6)$$

and hence

$$\mu = 2\mu^m. \quad (7)$$

The scaling in equation (7) does not alter the physical definition of the classical mass parameter μ . Rather, μ^m serves as a convenient analytical parameter for the rotating

dipole representation adopted in the present model.

Under the usual normalization of the restricted three-body problem described by [Szebehely & Grebenikov \(1969\)](#) and [Murray & Dermott \(1999\)](#),

$$m_1 + m_2 = 1. \quad (8)$$

Therefore,

$$m_1 = 1 - 2\mu^m, \quad m_2 = 2\mu^m, \quad (9)$$

and consequently,

$$m_{11} = m_{12} = \frac{1 - 2\mu^m}{2}, \quad m_2 = 2\mu^m. \quad (10)$$

It follows immediately that

$$m_{11} + m_{12} + m_2 = 1. \quad (11)$$

2.2 Barycentric Coordinates and Distance Functions

A uniformly rotating barycentric coordinate system (x, y, z) is adopted, with its origin located at the common center of mass of the two primaries. Let the centers of the larger and smaller primaries be located at $(x_1, 0, 0)$ and $(-x_2, 0, 0)$, respectively. The center-of-mass condition is therefore

$$m_1 x_1 - m_2 x_2 = 0. \quad (12)$$

Using the normalized masses gives

$$(1 - 2\mu^m)x_1 - (2\mu^m)x_2 = 0. \quad (13)$$

The characteristic distance between the two primaries is normalized to unity, so that

$$x_1 + x_2 = 1. \quad (14)$$

Substituting this condition into equation (12) yields

$$x_1 = 2\mu^m, \quad x_2 = 1 - 2\mu^m. \quad (15)$$

Since the larger primary consists of two equal dipole components separated by $2L$, their positions along the x -axis are

$$x_{11} = 2\mu^m - l, \quad x_{12} = 2\mu^m + l. \quad (16)$$

Thus, the two dipole masses are located at

$$P_{11} = (2\mu^m - l, 0), \quad P_{12} = (2\mu^m + l, 0), \quad (17)$$

while the triaxial secondary is located at

$$P_2 = (-1 + 2\mu^m, 0). \quad (18)$$

Let the infinitesimal body occupy the point $P(x, y)$ in the plane of motion. Its distances from the first dipole mass, second dipole mass, and triaxial secondary are respectively

$$r_{11}^2 = (x - 2\mu^m + l)^2 + y^2, \quad (19)$$

$$r_{12}^2 = (x - 2\mu^m - l)^2 + y^2, \quad (20)$$

and

$$r_2^2 = (x + 1 - 2\mu^m)^2 + y^2. \quad (21)$$

These distance functions determine the gravitational interaction of the infinitesimal body with the modified primaries.

2.3 Kinetic Energy in the Rotating Frame

Let the uniformly rotating coordinate system have angular velocity

$$\boldsymbol{\omega} = (0, 0, n), \quad (22)$$

where n denotes the mean motion of the primaries. The transformation between inertial and rotating velocities follows the standard rotating-frame kinematic relation discussed by [Szebehely & Grebenikov \(1969\)](#) and [Murray & Dermott \(1999\)](#).

The position vector of the infinitesimal particle is

$$\mathbf{r} = (x, y, 0), \quad (23)$$

while its velocity relative to the rotating frame is

$$\mathbf{V}_{\text{rot}} = (\dot{x}, \dot{y}, 0). \quad (24)$$

The corresponding inertial velocity is

$$\mathbf{V}_{\text{in}} = \mathbf{V}_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r}. \quad (25)$$

Since

$$\boldsymbol{\omega} \times \mathbf{r} = (-ny, nx, 0), \quad (26)$$

equation (25) gives

$$\mathbf{V}_{\text{in}} = (\dot{x} - ny, \dot{y} + nx, 0). \quad (27)$$

The kinetic energy of the infinitesimal body is therefore

$$T = \frac{1}{2}m [(\dot{x} - ny)^2 + (\dot{y} + nx)^2]. \quad (28)$$

Expanding equation (28),

$$\begin{aligned} T = & \frac{1}{2}m\dot{x}^2 - mn y \dot{x} + \frac{1}{2}mn^2 y^2 \\ & + \frac{1}{2}m\dot{y}^2 + mn x \dot{y} + \frac{1}{2}mn^2 x^2. \end{aligned} \quad (29)$$

Hence,

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mn(xy - yx) + \frac{1}{2}mn^2(x^2 + y^2). \quad (30)$$

For convenience, the kinetic energy is separated into three components,

$$T = T_2 + T_1 + T_0, \quad (31)$$

where

$$T_2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2), \quad (32)$$

$$T_1 = mn(xy - yx), \quad (33)$$

and

$$T_0 = \frac{1}{2}mn^2(x^2 + y^2). \quad (34)$$

Here, T_2 represents the ordinary translational kinetic energy, T_1 represents the velocity-dependent term associated with the rotation of the coordinate system, and T_0 represents

the centrifugal contribution.

2.4 Force-Ratio Parameter

According to Newton's law of universal gravitation, the magnitude of the gravitational force exerted by a primary mass m_i on the infinitesimal body is

$$f_g = \frac{Gmm_i}{d^2}, \quad (35)$$

where G is the universal gravitational constant, m is the mass of the infinitesimal body, m_i is the corresponding primary mass, and d denotes the relevant distance.

In the uniformly rotating frame, the corresponding centrifugal force is

$$f_c = mn^2d. \quad (36)$$

where n is the mean motion of the rotating system. The force-ratio parameter is introduced to control the strength of the gravitational contribution relative to the adopted rotational scaling and is defined as

$$k_i = \frac{f_g}{f_c}. \quad (37)$$

Substituting equations (35) and (36) gives

$$k_i = \frac{Gm_i}{n^2d^3}. \quad (38)$$

where $k = 1$ represents the usual Newtonian gravitational strength. Values $k > 1$ represent a stronger gravitational contribution, while $0 < k < 1$ represents a weaker gravitational contribution. This parameter provides a convenient measure of the gravitational contribution relative to the rotational contribution in the adopted nondimensional formulation.

2.5 Triaxial Secondary and Principal Moments of Inertia

The smaller primary m_2 is assumed to be a homogeneous triaxial ellipsoid whose semi-axes a , b , and c satisfy

$$a \neq b \neq c. \quad (39)$$

Its surface can be represented by

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} = 1. \quad (40)$$

For a homogeneous triaxial ellipsoid, the principal moments of inertia are given by [Murray & Dermott \(1999\)](#) and [Sharma et al. \(2001\)](#) as

$$\begin{aligned}
I_1 &= \frac{m_2}{5}(b^2 + c^2), \\
I_2 &= \frac{m_2}{5}(c^2 + a^2), \\
I_3 &= \frac{m_2}{5}(a^2 + b^2).
\end{aligned} \tag{41}$$

Let

$$\hat{\mathbf{r}}_2 = (\ell_1, \ell_2, \ell_3) \tag{42}$$

denote the unit vector directed from the center of the triaxial secondary to the infinitesimal particle. The direction cosines satisfy

$$\ell_1^2 + \ell_2^2 + \ell_3^2 = 1. \tag{43}$$

If the secondary is centered at $(x_2, 0, 0)$, then

$$\ell_1 = \frac{x - x_2}{r_2}, \quad \ell_2 = \frac{y}{r_2}, \quad \ell_3 = \frac{z}{r_2}. \tag{44}$$

The direction-dependent moment of inertia is therefore

$$I = I_1 \ell_1^2 + I_2 \ell_2^2 + I_3 \ell_3^2. \tag{45}$$

For the planar problem considered here,

$$z = 0, \tag{46}$$

and equation (45) becomes

$$I = I_1 \frac{(x - x_2)^2}{r_2^2} + I_2 \frac{y^2}{r_2^2}. \tag{47}$$

Table 1. Parameters used in the model

Parameter	Description
m_1	Mass of the rotating mass-dipole primary
m_2	Mass of the triaxial secondary
m_{11}, m_{12}	Equal mass components of the rotating mass dipole
μ	Classical mass parameter
μ^m	Scaled mass parameter
l	Dipole half-separation
R	Characteristic separation between the primary centres
a, b, c	Semi-axes of the triaxial secondary
σ_1	First triaxiality parameter
σ_2	Second triaxiality parameter
k	Force-ratio parameter
n	Mean motion of the primaries
G	Universal gravitational constant
r_{11}, r_{12}	Distances from the infinitesimal body to the dipole components
r_2	Distance from the infinitesimal body to the triaxial secondary

3 Results

3.1 Potential Energy of the Modified System

The total gravitational potential energy of the infinitesimal body is composed of the contributions from the two components of the rotating dipole, the Newtonian contribution of the secondary, and the correction arising from the triaxial geometry of the secondary. Accordingly,

$$V = V_{11} + V_{12} + V_{2n} + V_{\text{tri}}. \quad (48)$$

The Newtonian components are

$$V_{11} = -\frac{Gmm_{11}}{r_{11}}, \quad V_{12} = -\frac{Gmm_{12}}{r_{12}}, \quad V_{2n} = -\frac{Gmm_2}{r_2}. \quad (49)$$

Using the second-order exterior gravitational-potential correction of a rigid body expressed in terms of its principal moments of inertia, as discussed by [Szebehely & Grebenikov \(1969\)](#), [Murray & Dermott \(1999\)](#), and [Tricarico \(2008\)](#), the triaxial contribution is

$$V_{\text{tri}} = -\frac{Gm}{2r_2^3} (I_1 + I_2 + I_3 - 3I). \quad (50)$$

Substituting these contributions into equation (48) gives

$$V = -Gm \left[\frac{m_{11}}{r_{11}} + \frac{m_{12}}{r_{12}} + \frac{m_2}{r_2} + \frac{1}{2r_2^3} (I_1 + I_2 + I_3 - 3I) \right]. \quad (51)$$

Equation (51) represents the total potential energy of the infinitesimal body under the combined gravitational influence of the rotating dipole primary and the triaxial secondary.

From the force parameter defined in equation (38), taking $i = 1, 2$, one obtains

$$k_1 = \frac{Gm_1}{n^2d^3}, \quad (52)$$

and

$$k_2 = \frac{Gm_2}{n^2d^3}. \quad (53)$$

Adding equations (52) and (53) gives

$$k_1 + k_2 = \frac{G(m_1 + m_2)}{n^2d^3}. \quad (54)$$

Equivalently,

$$(k_1 + k_2)n^2d^3 = G(m_1 + m_2). \quad (55)$$

Let

$$k = k_1 + k_2, \quad (56)$$

where k denotes the total force-ratio parameter of the system. Equation (55) therefore becomes

$$kn^2d^3 = G(m_1 + m_2). \quad (57)$$

At this stage, the governing quantities are expressed in nondimensional form. Following the standard normalization used by [Szebehely & Grebenikov \(1969\)](#) and [Murray & Dermott \(1999\)](#), the characteristic distance between the two primary centers is taken as the unit distance,

$$d = 1, \quad (58)$$

the total mass is normalized according to

$$m_1 + m_2 = 1, \quad (59)$$

and time is scaled such that n represents the nondimensional angular velocity of the rotating frame.

Consequently,

$$d^3 = 1, \quad (60)$$

and equation (57) reduces to

$$G = kn^2. \quad (61)$$

Equation (61) represents the gravitational coefficient in the adopted nondimensional system. It does not imply that the dimensional Newtonian gravitational constant is numerically equal to kn^2 . Rather, the relation results from the adopted mass, distance, and time scales.

Substituting equation (61) into equation (51) gives

$$V = -kn^2m \left[\frac{m_{11}}{r_{11}} + \frac{m_{12}}{r_{12}} + \frac{m_2}{r_2} + \frac{1}{2r_2^3} (I_1 + I_2 + I_3 - 3I) \right]. \quad (62)$$

Using equation (41),

$$\begin{aligned} I_1 + I_2 + I_3 &= \frac{m_2}{5}(b^2 + c^2) + \frac{m_2}{5}(c^2 + a^2) \\ &\quad + \frac{m_2}{5}(a^2 + b^2), \end{aligned} \quad (63)$$

which simplifies to

$$I_1 + I_2 + I_3 = \frac{2m_2}{5}(a^2 + b^2 + c^2). \quad (64)$$

Similarly, from equation (45),

$$\begin{aligned} 3I &= 3 \left[\frac{m_2}{5}(b^2 + c^2) \frac{(x - x_2)^2}{r_2^2} \right. \\ &\quad \left. + \frac{m_2}{5}(c^2 + a^2) \frac{y^2}{r_2^2} + \frac{m_2}{5}(a^2 + b^2) \frac{z^2}{r_2^2} \right]. \end{aligned} \quad (65)$$

Since the motion is planar, $z = 0$, and hence

$$3I = \frac{3m_2}{5r_2^2} [(b^2 + c^2)(x - x_2)^2 + (c^2 + a^2)y^2]. \quad (66)$$

Substituting equations (64) and (66) into equation (62) yields

$$\begin{aligned}
V = -kn^2m \left[\frac{m_{11}}{r_{11}} + \frac{m_{12}}{r_{12}} + \frac{m_2}{r_2} \right. \\
\left. + \frac{1}{2r_2^3} \left\{ \frac{2m_2}{5}(a^2 + b^2 + c^2) \right. \right. \\
\left. \left. - \frac{3m_2}{5r_2^2} [(b^2 + c^2)(x - x_2)^2 + (c^2 + a^2)y^2] \right\} \right]. \quad (67)
\end{aligned}$$

From equation (21),

$$r_2^2 = (x - x_2)^2 + y^2, \quad (68)$$

and therefore

$$(x - x_2)^2 = r_2^2 - y^2. \quad (69)$$

Substituting equation (69) into equation (67) gives

$$\begin{aligned}
V = -kn^2m \left[\frac{m_{11}}{r_{11}} + \frac{m_{12}}{r_{12}} + \frac{m_2}{r_2} \right. \\
\left. + \frac{1}{2r_2^3} \left\{ \frac{2m_2}{5}(a^2 + b^2 + c^2) \right. \right. \\
\left. \left. - \frac{3m_2}{5r_2^2} [(b^2 + c^2)(r_2^2 - y^2) + (c^2 + a^2)y^2] \right\} \right]. \quad (70)
\end{aligned}$$

Upon collecting similar terms, the potential becomes

$$\begin{aligned}
V = -kn^2m \left[\frac{m_{11}}{r_{11}} + \frac{m_{12}}{r_{12}} + \frac{m_2}{r_2} \right. \\
\left. + \frac{m_2}{10r_2^3} \left(2a^2 - b^2 - c^2 + \frac{3(b^2 - a^2)y^2}{r_2^2} \right) \right]. \quad (71)
\end{aligned}$$

Equivalently,

$$V = -kn^2m \left[\frac{m_{11}}{r_{11}} + \frac{m_{12}}{r_{12}} + \frac{m_2}{r_2} + \frac{m_2}{2r_2^3} \left(\frac{2a^2 - b^2 - c^2}{5} + \frac{3(b^2 - a^2)y^2}{5r_2^2} \right) \right]. \quad (72)$$

Following the triaxial parametrization used by Sharma et al. (2003), the triaxiality parameters are introduced as

$$\sigma_1 = \frac{a^2 - c^2}{5R^2}, \quad \sigma_2 = \frac{b^2 - c^2}{5R^2}, \quad (73)$$

where σ_1 and σ_2 denote the first and second triaxiality parameters, respectively, and R is the characteristic separation between the primary centers.

From equation (73),

$$2a^2 - b^2 - c^2 = 5R^2(2\sigma_1 - \sigma_2), \quad (74)$$

while

$$b^2 - a^2 = -5R^2(\sigma_1 - \sigma_2). \quad (75)$$

Under the adopted normalization $R = 1$, substituting equations (74) and (75) into equation (72) gives

$$V = -kn^2m \left[\frac{m_{11}}{r_{11}} + \frac{m_{12}}{r_{12}} + \frac{m_2}{r_2} + \frac{m_2(2\sigma_1 - \sigma_2)}{2r_2^3} - \frac{3m_2(\sigma_1 - \sigma_2)y^2}{2r_2^5} \right]. \quad (76)$$

Finally, substituting the normalized masses from equation (10) gives

$$V = -kn^2m \left[\frac{1 - 2\mu^m}{2r_{11}} + \frac{1 - 2\mu^m}{2r_{12}} + \frac{2\mu^m}{r_2} + \frac{\mu^m(2\sigma_1 - \sigma_2)}{r_2^3} - \frac{3\mu^m(\sigma_1 - \sigma_2)y^2}{r_2^5} \right]. \quad (77)$$

3.2 Mean Motion

Let R denote the distance between the centre of the rotating mass-dipole primary m_1 and the triaxial secondary m_2 . Since the primary consists of two equal masses separated by a distance $2l$, the distances of the two dipole components from the secondary are

$$d_{11} = R - l, \quad d_{12} = R + l, \quad (78)$$

where $0 \leq l < R$.

Let

$$\Delta = 2\sigma_1 - \sigma_2. \quad (79)$$

For motion along the line joining the primary centres, the interaction potential between each dipole component and the triaxial secondary contains the Newtonian term and the triaxial correction. Since

$$m_{11} = m_{12} = \frac{m_1}{2}, \quad (80)$$

the gravitational force between the two primary bodies is

$$F = \frac{Gm_1m_2}{2} \left[\frac{1}{(R-l)^2} + \frac{1}{(R+l)^2} \right] + \frac{3Gm_1m_2R^2\Delta}{4} \left[\frac{1}{(R-l)^4} + \frac{1}{(R+l)^4} \right]. \quad (81)$$

Let t_1 and t_2 denote the distances of the centres of m_1 and m_2 , respectively, from the common centre of mass. For circular motion,

$$m_1n^2t_1 = F, \quad (82)$$

and

$$m_2n^2t_2 = F. \quad (83)$$

Dividing by the respective masses and adding gives

$$n^2(t_1 + t_2) = F \left(\frac{1}{m_1} + \frac{1}{m_2} \right). \quad (84)$$

Since

$$t_1 + t_2 = R, \quad (85)$$

we obtain

$$n^2 = \frac{G(m_1 + m_2)}{2R} \left[\frac{1}{(R-l)^2} + \frac{1}{(R+l)^2} + \frac{3R^2\Delta}{2} \left(\frac{1}{(R-l)^4} + \frac{1}{(R+l)^4} \right) \right]. \quad (86)$$

Under the usual normalization

$$R = 1, \quad m_1 + m_2 = 1, \quad G = 1, \quad (87)$$

equation (86) becomes

$$n^2 = \frac{1}{2} \left[\frac{1}{(1-l)^2} + \frac{1}{(1+l)^2} \right] + \frac{3\Delta}{4} \left[\frac{1}{(1-l)^4} + \frac{1}{(1+l)^4} \right]. \quad (88)$$

After simplification,

$$n^2 = \frac{1+l^2}{(1-l^2)^2} + \frac{3}{2}(2\sigma_1 - \sigma_2) \frac{1+6l^2+l^4}{(1-l^2)^4}. \quad (89)$$

Therefore,

$$n = \left[\frac{1+l^2}{(1-l^2)^2} + \frac{3}{2}(2\sigma_1 - \sigma_2) \frac{1+6l^2+l^4}{(1-l^2)^4} \right]^{1/2}. \quad (90)$$

For n to be real, the expression in equation (89) must be positive. Thus,

$$2\sigma_1 - \sigma_2 > -\frac{2}{3} \frac{(1+l^2)(1-l^2)^2}{1+6l^2+l^4}. \quad (91)$$

When $l = 0$, equation (89) reduces to

$$n^2 = 1 + \frac{3}{2}(2\sigma_1 - \sigma_2), \quad (92)$$

which is the mean-motion expression obtained when the rotating mass dipole reduces to a single mass located at the centre of the larger primary. Hence, the dipole half-separation l appears explicitly in the general mean-motion expression and the earlier result is recovered as the limiting case $l = 0$.

3.3 Hamiltonian Formulation and Conjugate Momenta

The Hamiltonian formulation is employed to obtain the equations of motion in canonical form. The approach follows the rotating-frame canonical formulation discussed by [Szebehely & Grebenikov \(1969\)](#) and [Sharma et al. \(2001\)](#).

Using the decomposition of the kinetic energy, the Hamiltonian is written as

$$H = T_2 - T_0 + V. \quad (93)$$

Hence,

$$H = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}mn^2(x^2 + y^2) + V. \quad (94)$$

The conjugate momenta corresponding to x and y are

$$p_x = m\dot{x} - mny, \quad p_y = m\dot{y} + mnx. \quad (95)$$

Solving equation (95) for the velocity components gives

$$\dot{x} = \frac{p_x}{m} + ny, \quad \dot{y} = \frac{p_y}{m} - nx. \quad (96)$$

Substituting equation (96) into equation (94) gives

$$H = \frac{1}{2}m \left[\left(\frac{p_x}{m} + ny \right)^2 + \left(\frac{p_y}{m} - nx \right)^2 \right] - \frac{1}{2}mn^2(x^2 + y^2) + V. \quad (97)$$

Expanding,

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + nyp_x - nxp_y + \frac{1}{2}mn^2(x^2 + y^2) - \frac{1}{2}mn^2(x^2 + y^2) + V. \quad (98)$$

The rotational quadratic terms cancel, yielding

$$H = \frac{p_x^2 + p_y^2}{2m} + n(yp_x - xp_y) + V. \quad (99)$$

Hamilton's canonical equations are

$$\dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{y} = \frac{\partial H}{\partial p_y}, \quad (100)$$

and

$$\dot{p}_x = -\frac{\partial H}{\partial x}, \quad \dot{p}_y = -\frac{\partial H}{\partial y}. \quad (101)$$

From equations (99) and (100),

$$\dot{x} = \frac{p_x}{m} + ny, \quad \dot{y} = \frac{p_y}{m} - nx. \quad (102)$$

Similarly,

$$\dot{p}_x = np_y - \frac{\partial V}{\partial x}, \quad \dot{p}_y = -np_x - \frac{\partial V}{\partial y}. \quad (103)$$

Differentiating equation (95) with respect to time gives

$$\dot{p}_x = m\ddot{x} - mn\dot{y}, \quad \dot{p}_y = m\ddot{y} + mn\dot{x}. \quad (104)$$

Combining equations (103) and (104), one obtains

$$\ddot{x} - 2n\dot{y} = n^2x - \frac{1}{m} \frac{\partial V}{\partial x}, \quad (105)$$

and

$$\ddot{y} + 2n\dot{x} = n^2 y - \frac{1}{m} \frac{\partial V}{\partial y}. \quad (106)$$

3.4 Effective Potential and Governing Equations

Define the gravitational contribution per unit mass as

$$F(x, y) = -\frac{V}{m}. \quad (107)$$

Using equation (77),

$$F(x, y) = kn^2 \left[\frac{1 - 2\mu^m}{2r_{11}} + \frac{1 - 2\mu^m}{2r_{12}} + \frac{2\mu^m}{r_2} + \frac{\mu^m(2\sigma_1 - \sigma_2)}{r_2^3} - \frac{3\mu^m(\sigma_1 - \sigma_2)y^2}{r_2^5} \right]. \quad (108)$$

The effective potential is therefore

$$\Omega(x, y) = \frac{1}{2}n^2(x^2 + y^2) + F(x, y). \quad (109)$$

Hence,

$$\begin{aligned} \Omega(x, y) = & \frac{1}{2}n^2(x^2 + y^2) \\ & + kn^2 \left[\frac{1 - 2\mu^m}{2r_{11}} + \frac{1 - 2\mu^m}{2r_{12}} + \frac{2\mu^m}{r_2} + \frac{\mu^m(2\sigma_1 - \sigma_2)}{r_2^3} - \frac{3\mu^m(\sigma_1 - \sigma_2)y^2}{r_2^5} \right]. \end{aligned} \quad (110)$$

The equations of motion can consequently be expressed in the compact form

$$\ddot{x} - 2n\dot{y} = \Omega_x, \quad (111)$$

$$\ddot{y} + 2n\dot{x} = \Omega_y, \quad (112)$$

where

$$\Omega_x = \frac{\partial \Omega}{\partial x}, \quad \Omega_y = \frac{\partial \Omega}{\partial y}. \quad (113)$$

Equations (19)–(21), (90), (99), (110), (111), and (112) constitute the governing mathematical formulation of the modified restricted three-body problem involving a rotating mass-dipole primary and a triaxial secondary.

4 Discussion

The results show that the rotating mass-dipole primary and the triaxial secondary both contribute to the dynamics of the infinitesimal body. The effective potential and equations of motion show how the mass parameter, dipole separation, force-ratio parameter, and triaxiality parameters enter the system. In particular, the triaxiality of the secondary changes the gravitational potential and also contributes to the mean motion. This agrees with earlier studies showing that the shape and mass distribution of celestial bodies can affect their dynamical behaviour [Abozaid et al. \(2024\)](#); [Radwan & Abd El-Motelp \(2021\)](#); [Zeng et al. \(2015\)](#).

The Hamiltonian and effective potential obtained in the study provide the mathematical basis for examining the motion of particles within the system. The equations of motion also show that treating the primaries as non-spherical bodies gives a broader model than the classical point-mass case. However, the present study is limited to the mathematical formulation of the model. Further studies can use the developed equations to investigate equilibrium points, their stability, periodic orbits, and particle trajectories under different parameter values.

5 Conclusion

This study developed a modified restricted three-body model in which the larger primary is represented by a rotating mass dipole and the smaller primary is considered as a homogeneous triaxial body. The model combines the effects of asymmetric mass distribution and triaxiality within a single mathematical framework.

The kinetic and potential energies of the infinitesimal body were formulated in a uniformly rotating coordinate system. The mean motion, Hamiltonian, conjugate momenta, effective potential, and equations of motion were also derived. The resulting equations show how the dipole half-separation, force-ratio parameter, mass parameter, and triaxiality parameters enter the dynamics of the system.

The developed model extends the classical restricted three-body problem by considering primaries with more realistic mass distributions and shapes. The formulation can be used as a basis for further studies of equilibrium points, stability, periodic orbits, and the motion of particles in binary systems containing irregular celestial bodies.

Availability of data and material

Not Applicable.

Competing Interest

No competing interest is hereby declared by the authors.

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Authors' contributions

The development of the model, analysis, discussion of results and typesetting were implemented by all the authors and was approved for final manuscript.

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