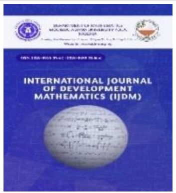
International Journal of Development Mathematics Vol 1 Issue 2, Page No. 024 - 037

INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS

ISSN: 3026-8656 (Print) | 3026-8699 (Online)

journal homepage: https://ijdm.org.ng/index.php/Journals



Mathematical Model of the Dynamics of three Different Species in Predator-Prey System

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ARTICLE INFO

Article history:
Received 28 December 2023
Received in revised form 17 March 2024
Accepted 27 April 2024

Keywords:
Differential equations, Positively – Invariant, Predator-Prey, Stability.

MSC 2020 Subject classification:
34A34, 92B05, 92-10

ABSTRACT

To understand how organisms interact with themselves and their surroundings, it is necessary to use mathematical models to explore the science of multispecies cohabitation. This paper studied the interaction between three species of animals (Lion, Leopard, and Hare) in an environment using a formulated predator-prey model of system of three nonlinear differential equations of first order with five equilibrium points. The equilibrium points were found using the Routh Hurwitz Stability Criterion to be locally asymptotically stable. It was also established that the solution of the model exist and are all positive for any $t \geqslant 0$. Some numerical simulations were carried out to show the analytical solution of the model and the results presented. The results show that the predator-prey presented in this work can be used perfectly to study any species interaction with the same nature with the ones considered in this paper. For further studies, this model can be extended to include four or more species interactions.

1. Introduction

Over time, mathematical models have been utilised to comprehend complex situations in natural sciences and engineering (Kwaghkor, 2020). These mathematical models can either be deterministic (Kwaghkor and Luga, 2016; Kwaghkor *et al.,* 2024) or stochastic (Kwaghkor *et al.,* 2018; Kwaghkor *et al.*, 2019); Kwaghkor *et al.*, 2021) and are based on ordinary differential equations (Adamu, 2023, Alkali *et al*., 2023; Kwaghkor *et al*., 2022), partial differential equations (Orapine *et al.*, 2023) and Stochastic differential equations (Kwaghkor *et al.*, 2021; Kwaghkor, 2022). In studying the interrelationships of organisms and their environment, there is need to investigate science of coexistence of two or more species. To this end, it is natural to seek a mathematical formulation of this prey-predator problem and to use it to forecast the behavior of populations of various species at different times (Vahidin, *et al*., 2017; Ma *et al*., 2017).

Nonlinear differential equations are utilized in the study of Lotka-Volterra prey-predator relationships (Canale, 1970). Mathematical models of the interaction between predator and prey populations are generally expressed as systems of nonlinear ordinary differential equations (Bai and Zhang, 2022; Canale, 1970; Xu and Wu, 2013). In animal ecosystems, interspecies interaction is inevitable (Ashine and Gebru, 2017). Interactions between various species occur on a regular basis when they live in comparable habitats. By offering havens, the natural world can offer a certain level of defense to prey populations (Ashine and Gebru, 2017). Such refugia can help in prolonging prey- predator interactions by reducing the chance of extinction due to predation (Ashine and Gebru, 2017).

Italian mathematician Vito Volterra developed a differential equation model in the 1920s to explain the population dynamics of a predator and its prey (Vahidin *et al*., 2017; Xia and Cao, 2006). Predators can sustain a higher population if there is a huge number of prey. But when the number of predators becomes too much, the prey starts to disappear, which likewise causes the number of predators to decline (Vahidin *et al*., 2017).

Vahidin *et al*. (2017) studied the system of differential equations modeling the population dynamics of a predator y , a scavenger z , and a prey x to demonstrate the possible population trends when a predator, a prey and a scavenger population interact. Their study shows that, the predator and the prey can coexist in the absence of the scavenger, and the scavenger and the prey can coexist in the absence of the predator. However, the scavenger and the predator cannot coexist without the prey. Biologically, this is reasonable, because without the prey, the predator will have no

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https://doi.org/10.62054/ijdm/0102.03

food and will die of hunger. The scavenger will then lose all sources of food and will die of hunger too. It has been shown also that all the three populations can coexist in two ways: they will oscillate between stable populations over time, or the populations will oscillate until they saturate and remain constant over time (Hadžiabdić *et al.*, 2017).

In order to study the population dynamics of a three-species prey-predator, Hadžiabdić *et al*. (2017) considered a simplified model from the Lotka-Volterra model which involves one prey and two predators which was described by a system of three nonlinear equations of first order. Their study shows that the system has three equilibrium points. The study of the coexistence shows that, if prey $x = 0$, then predators y and z are dying and if for predators y and z hold $y = z = 0$, then the prey x tends to infinity. As mentioned earlier, the model by Hadžiabdić *et al*. (2017) considered a situation where two predators are depending on one prey.

This study intends to modify the Hadžiabdić *et al*. (2017) by considering a situation in nature where one species of animal feeds on another species of animal, which in turn feeds on other species of animals. The first specie is called the first predator, the second is called the second predator/first prey and the third is called the second prey which is the consideration in this paper. This paper therefore studied the interaction between three species of animals (Lion, Leopard and Hare) in an environment. Lions are carnivores, and some of the type of prey they catch includes but not limited to Leopard and Hares. Leopards are carnivores, and some of the type of prey they catch include but not limited to Hares. Hares are herbivores, which means they feed mainly on plants. In this study, Lions are considered as the population of the first predator, Leopards are considered as the population of the second predator/first prey and Hares are considered as the population of the second prey.

2. Methodology

In order to create a mathematical model which describes the relationship between predators and the preys, few assumptions are made as presented below.

2.1 Assumptions of the Model

i. The first specie is predator, the second specie is prey to the first specie and predator to the third specie, while the third specie is prey to both the first and second predators.
ii. The first predator only dies from natural causes.
iii. The second predator/first prey only die by being eaten by the first predator, and of natural causes.
iv. The second prey only die by being eaten by the first predator, second predator and of natural causes.
v. The interactions between predators and the preys are described by functions.
vi. The Model equations developed in this research describe the dynamics of biological systems of non-linear differential equations which focuses only on the predator-prey interactions and ignores competition, disease, and mutualism.

2.2 The Lion-Leopard-Hare interaction model

The nonlinear system is given by

$$\frac{dx}{dt} = \alpha x + \delta xy + \rho xz - \sigma x \tag{1}$$

$$\frac{dy}{dt} = \beta y + \tau yz - \varphi xy - \kappa y \tag{2}$$

$$\frac{dz}{dt} = \gamma z - \mu xz - \varepsilon yz - \xi z \tag{3}$$

where $x(0) = x_0$, $y(0) = y_0$ and $\mathrm{z}(0) = z_0$

$x(t)$ represents the population of the Lions at time $t, y(t)$ represents the population of the Leopards at time $t, z(t)$ represents the population of the Hares at time t, x_0 is the initial size of the Lions population, y_0 is the initial size of the Leopards population and z_0 is the initial size of the Hares population.

2.2 Description of the model

Considering Equation (1), $x'(t)$ which is the rate of change of the Lion population with respect to time t, is described by four different terms. It is negatively influenced by natural death rate, $-\sigma x$, of Lion population. It is positively influenced by the natural birth rate of the Lion population, Lion-Leopard interaction and Leopard-Hares interaction by the terms αx, δxy and ρxz respectively. α, δ, ρ and σ are constants non-negative real number as described in Table 1.

Considering Equation (2), $y'(t)$ which is the rate of change of the Leopard population with respect to time t, is described by four different terms. It is negatively influenced by the natural death rate of Leopard population and Lion-Leopard interaction by the terms $-\kappa y$ and $-\varphi xy$ respectively. It is positively influenced by the natural birth rate of the Leopard and Leopard-Hares interaction as shown by the terms βy and τyz respectively. β, τ φ and κ are constants non-negative real number as described in Table 1.

Considering Equation (3), $z'(t)$ which is the rate of change of the Hare population with respect to time t, is described by four different terms. It is positively influenced by natural birth of the Hares population, as shown by the term γz. It is negatively influenced by the natural death rate of the Leopard, Lion-Hares interaction and the Leopard-Hares interaction as shown by the terms $-\xi z$, $-\mu xz$ and $-\varepsilon yz$ respectively. γ, μ, ε and ξ are non-negative real numbers as described in Table 1.

Table 1: Variables and Parameters of the Model

Variables/Parameters	Definitions
$x(t)$	Population of the Lions at time t
$y(t)$	Population of the Leopards at time t
$z(t)$	Population of the Hares at time t
α	The growth rate of x
σ	The natural death rate of x
δ	The rate of change of x due to the presence of y
ρ	The rate of change of the x due to the presence of z
β	The growth rate of y
κ	The natural death rate of y
τ	The rate of change of y due to the presence of z
φ	The rate of change of y due to the presence of x
γ	The growth rate of z
ξ	The natural death rate of z
μ	The rate of change of z due to the presence of x
ε	The rate of change of z due to the presence of y

2.3 Model Analysis

2.3.1 Positivity of solution

For the model equation (1) - (3) to have biological meaning, the solutions $x(t)$, $y(t)$ and $z(t)$ must all be positive, hence the need for the positivity analysis.

Theorem 1: The region $\theta = \{(x, y, z) \in \mathrm{R}_{+}^{3} \ : \ x \geq 0, y \geq 0, z \geq 0\}$ is positively-invariant for the model (1) - (3) with the solutions x(t), y(t) and z(t) all positive for $t \geq 0.$

Proof: To prove the theorem, we show by the method of contradiction in Bhunu *et al.*, (2009) that, the following three cases are contradicted.

Case 1: If there exist $t_1 > 0$ such that $x(t_1) = 0, x'(t_1) < 0, y(t) > 0, z(t) > 0$ for $0 < t < t_1$ or

Case 2: If there exist $t_2 > 0$ such that $y(t_2) = 0, y'(t_2) < 0, x(t) > 0, z(t) > 0$ for $0 < t < t_2$ or

Case 3: If there exist $t_3 > 0$ such that $z(t_3) = 0, z'(t_3) < 0, x(t) > 0, y(t) > 0$ for $0 < t < t_3$.

For case 1: $\dfrac{dx}{dt} = \alpha x + \delta xy + \rho xz - \sigma x = 0$ which is not negative. Hence, the contradiction meaning that the solution $x(t)$ remains positive.

For case 2: $\dfrac{dy}{dt} = \beta y + \tau yz - \varphi xy - \kappa y = 0$ which is not negative. Hence, the contradiction meaning that the solution $y(t)$ remains positive.

For case 3: $\dfrac{dy}{dt} = \gamma z - \mu xz - \varepsilon yz - \xi z = 0$ which is not negative. Hence, the contradiction meaning that the solution $z(t)$ remains positive.

So in all the cases, the solution $x(t), y(t)$ and $z(t)$ in equation (1) - (3) remain positive. Hence, the proof.

2.3.2 Equilibrium points of the Model

The model has five (5) positive equilibrium points as follows. At equilibrium $x'(t) = y'(t) = z'(t) = 0.$

First Equilibrium Point (EP_1): $(x_1, y_1, z_1) = (0, 0, 0)$,

Second Equilibrium Point (EP_2): $(x_2, y_2, z_2) = \left(0, \frac{\gamma-\xi}{\varepsilon}, \frac{\kappa-\beta}{\tau}\right)$,

The third Equilibrium Point (EP_3): $(x_3, y_3, z_3) = \left(\frac{\gamma-\xi}{\mu}, 0, \frac{\sigma-\alpha}{\rho}\right)$,

The fourth Equilibrium Point (EP_4): $(x_4, y_4, z_4) = \left(\frac{\beta-\kappa}{\varphi}, \frac{\sigma-\alpha}{\delta}, 0\right)$,

The fifth Equilibrium Point (EP_5): $(x_5, y_5, z_5) = (x^*, y^*, z^*)$

where $x^* = \left(\frac{1}{\tau\delta+\rho\varphi}\right)(\sigma\tau - \alpha\tau - \kappa\rho + \beta\rho)$, $y^* = \left(\frac{1}{\varepsilon(\tau\delta+\rho\varphi)}\right)(\alpha\tau\mu - \sigma\tau\mu + \tau\gamma\delta - \tau\delta\xi + \kappa\mu\rho - \beta\mu\rho + \gamma\rho\varphi - \xi\rho\varphi)$ and $z^* = \left(\frac{1}{\tau\delta+\rho\varphi}\right)(\kappa\delta - \beta\delta - \alpha\varphi + \sigma\varphi)$.

2.3.3 Local Stability of the Equilibrium Points

Theorem 2: (**Routh – Hurwitz Stability Criterion**): Given the polynomial of positive real constant coefficients

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n \tag{4}$$

with the Hurwitz Matrices

$$H_j = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & a_n \end{pmatrix}, \quad j = 1, 2, \dots, n \tag{5}$$

All the roots of the polynomial (4) are negative or have negative real part (which is indicating locally asymptotic stability) if and only if the determinant of all the Hurwitz Matices are positive (That is $detH_j > 0, j = 1, 2, \dots, n$).

Corollary 1: When $n = 3$, the polynomial (4) becomes

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \tag{6}$$

with the Hurwitz Matrix

$$H_1 = (a_1), \qquad H_2 = \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & 1 \\ 0 & 0 & a_3 \end{pmatrix} \tag{7}$$

All the roots of the polynomial (4) are negative or have negative real part (which is indicating locally asymptotic stability) if and only if the determinant of all the Hurwitz Matices are positive. That is $detH_1 = a_1 > 0$, $detH_2 = a_1a_2 > 0$ and $detH_3 = a_1a_2 > a_3$.

Proof: The proof of corollary 1 implies the proof of Theorem 2. So the proof of corollary 1 is done by taking the Jacobian of the system (1) - (3) as presented below

$$J_{(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{pmatrix} \alpha - \sigma + \delta y + \rho z & \delta x & \rho x \\ -\varphi y & \beta - \kappa - \varphi y + \tau z & \tau y \\ -\mu z & -\varepsilon z & \gamma - \xi - \mu x - \varepsilon y \end{pmatrix} \tag{8}$$

The Jacobian at EP_1, EP_2, EP_3, EP_4 and EP_5 are presented as

$$J_{EP_1} = \begin{pmatrix} \alpha - \sigma & 0 & 0 \\ 0 & \beta - \kappa & 0 \\ 0 & 0 & \gamma - \xi \end{pmatrix} \tag{9}$$

$$J_{EP_2} = \begin{pmatrix} \frac{\kappa\varepsilon\rho - \beta\varepsilon\rho + \tau\varepsilon\alpha - \tau\varepsilon\sigma + \tau\gamma\delta - \tau\xi\delta}{\tau\varepsilon} & 0 & 0 \\ \frac{\varphi\xi - \varphi\gamma}{\varepsilon} & 0 & \frac{\tau\gamma - \tau\xi}{\varepsilon} \\ \frac{\beta\mu - \kappa\mu}{\tau} & \frac{\beta\varepsilon - \kappa\varepsilon}{\tau} & 0 \end{pmatrix} \tag{10}$$

$$J_{EP_3} = \begin{pmatrix} 0 & \frac{\gamma\delta - \xi\delta}{\mu} & \frac{\gamma\rho - \xi\rho}{\mu} \\ 0 & \frac{\sigma\tau\mu - \alpha\tau\mu - \kappa\mu\rho + \beta\mu\rho - \gamma\rho\varphi + \xi\rho\varphi}{\mu\rho} & 0 \\ \frac{\alpha\mu - \sigma\mu}{\rho} & \frac{\alpha\varepsilon - \sigma\varepsilon}{\rho} & 0 \end{pmatrix} \tag{11}$$

$$J_{EP_4} = \begin{pmatrix} 0 & \frac{\sigma\beta - \sigma\kappa}{\varphi} & \frac{\rho\beta - \rho\kappa}{\varphi} \\ \frac{\varphi\alpha - \varphi\sigma}{\delta} & 0 & \frac{\sigma\tau - \alpha\tau}{\delta} \\ 0 & 0 & \frac{\kappa\mu\delta - \beta\mu\delta + \alpha\varepsilon\varphi - \sigma\varepsilon\varphi + \gamma\delta\varphi - \delta\xi\varphi}{\delta\varphi} \end{pmatrix} \tag{12}$$

$$J_{EP_5} = \begin{pmatrix} \alpha - \sigma + \delta y^* + \rho z^* & \delta x^* & \rho x^* \\ -\varphi y^* & \beta - \kappa - \varphi y^* + \tau z^* & \tau y^* \\ -\mu z^* & -\varepsilon z^* & \gamma - \xi - \mu x^* - \varepsilon y^* \end{pmatrix} \tag{13}$$

The characteristic polynomials for equations (9) - (13) using $|J_{EP} - \lambda I| = 0$ are respectively presented below.

From equation (9)

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{14}$$

where

$a_1 = \kappa - \alpha + \sigma - \beta - \gamma + \xi > 0,$

$a_2 = \kappa\sigma - \alpha\kappa + \alpha\beta - \sigma\beta + \alpha\gamma - \kappa\gamma - \sigma\gamma + \beta\gamma - \alpha\xi + \kappa\xi + \sigma\xi - \beta\xi > 0$

$a_3 = \alpha\kappa\gamma - \kappa\sigma\gamma - \alpha\beta\gamma + \sigma\beta\gamma - \alpha\kappa\xi + \kappa\sigma\xi + \alpha\beta\xi - \sigma\beta\,\xi > 0.$

From equation (10),

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \tag{15}$$

where

$b_1 = \sigma\varepsilon^2 - \alpha\varepsilon^2 - \gamma\delta\varepsilon + \delta\varepsilon\xi - \rho\left(\frac{\kappa}{\tau}\right)\varepsilon^2 + \rho\left(\frac{\beta}{\tau}\right)\varepsilon^2 > 0,$

$b_2 = \kappa\gamma - \beta\gamma - \kappa\xi + \beta\xi > 0$

$$\begin{aligned} b_3 = {} & \kappa\sigma\gamma\varepsilon^2 - \alpha\kappa\gamma\varepsilon^2 + \alpha\beta\gamma\varepsilon^2 - \sigma\beta\gamma\varepsilon^2 + \alpha\kappa\varepsilon^2\xi - \kappa\sigma\varepsilon^2\xi - \alpha\beta\varepsilon^2\xi - \kappa\gamma^2\delta\varepsilon + \sigma\beta\varepsilon^2\xi + \beta\gamma^2\delta\varepsilon \\ & - \kappa\,\delta\varepsilon\xi^2 + \beta\delta\varepsilon\xi^2 + 2\kappa\gamma\delta\varepsilon\xi - 2\beta\gamma\delta\varepsilon\xi - \rho\left(\frac{\kappa^2}{\tau}\right)\gamma\varepsilon^2 - \rho\left(\frac{\beta^2}{\tau}\right)\gamma\varepsilon^2 + \rho\left(\frac{\kappa^2}{\tau}\right)\varepsilon^2\xi \\ & + \rho\left(\frac{\beta^2}{\tau}\right)\varepsilon^2\xi + 2\rho\kappa\left(\frac{\beta}{\tau}\right)\gamma\varepsilon^2 - 2\rho\,\kappa\left(\frac{\beta}{\tau}\right)\varepsilon^2\xi > 0. \end{aligned}$$

From equation (11),

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0 \tag{16}$$

where

$$c_1 = \rho^2\kappa - \rho^2\beta + \rho\alpha\tau - \rho\sigma\tau + \rho^2\left(\frac{\gamma}{\mu}\right)\varphi - \left(\frac{\rho^2}{\mu}\right)\varphi\xi > 0,$$

$$c_2 = \sigma\gamma - \alpha\gamma + \alpha\xi - \sigma\xi > 0,$$

$$\begin{aligned} c_3 = {} & \rho^2\alpha\kappa\gamma - \rho^2\kappa\sigma\gamma + \rho\alpha^2\tau\gamma + \rho\sigma^2\tau\gamma - \rho^2\alpha\beta\gamma + \rho^2\sigma\beta\gamma - \rho^2\alpha\kappa\xi + \rho^2\kappa\sigma\xi - \rho\alpha^2\tau\xi - \rho\sigma^2\tau\xi \\ & + \rho^2\alpha\beta\xi - \rho^2\sigma\beta\xi - 2\rho\alpha\sigma\tau\gamma + 2\rho\alpha\sigma\tau\xi + \rho^2\alpha\left(\frac{\gamma^2}{\mu}\right)\varphi - \rho^2\sigma\left(\frac{\gamma^2}{\mu}\right)\varphi \\ & + \rho^2\left(\frac{\alpha}{\mu}\right)\varphi\xi^2 - \rho^2\left(\frac{\sigma}{\mu}\right)\varphi\xi^2 - 2\rho^2\alpha\left(\frac{\gamma}{\mu}\right)\varphi\xi + 2\rho^2\sigma\left(\frac{\gamma}{\mu}\right)\varphi\xi > 0. \end{aligned}$$

From equation (12),

$$\lambda^3 + d_1\lambda^2 + d_2\lambda + d_3 = 0 \tag{17}$$

where

$$d_1 = \varphi^2\xi - \gamma\varphi^2 - \kappa\mu\varphi + \beta\mu\varphi - \left(\frac{\alpha}{\delta}\right)\varphi^2\varepsilon + \left(\frac{\sigma}{\delta}\right)\varphi^2\varepsilon > 0,$$

$$d_2 = \sigma^2\left(\frac{\beta}{\delta}\right) - \kappa\left(\frac{\sigma^2}{\delta}\right) + \alpha\kappa\left(\frac{\sigma}{\delta}\right) - \alpha\sigma\left(\frac{\beta}{\delta}\right) > 0,$$

$$\begin{aligned} d_3 = {} & \kappa\sigma^2\left(\frac{\gamma}{\delta}\right)\varphi^2 + \kappa^2\sigma^2\left(\frac{\mu}{\delta}\right)\varphi - \sigma^2\beta\left(\frac{\gamma}{\delta}\right)\varphi^2 + \sigma^2\beta^2\left(\frac{\mu}{\delta}\right)\varphi - \kappa\left(\frac{\sigma^2}{\delta}\right)\varphi^2\xi + \sigma^2\left(\frac{\beta}{\delta}\right)\varphi^2\xi \\ & - \kappa\left(\frac{\sigma^3}{\delta^2}\right)\varphi^2\varepsilon + \sigma^3\left(\frac{\beta}{\delta^2}\right)\varphi^2\varepsilon - \alpha\kappa\sigma\left(\frac{\gamma}{\delta}\right)\varphi^2 - \alpha\kappa^2\sigma\left(\frac{\mu}{\delta}\right)\varphi + \alpha\sigma\beta\left(\frac{\gamma}{\delta}\right)\varphi^2 \\ & - \alpha\sigma\beta^2\left(\frac{\mu}{\delta}\right)\varphi - 2\kappa\sigma^2\beta\left(\frac{\mu}{\delta}\right)\varphi + \alpha\kappa\left(\frac{\sigma}{\delta}\right)\varphi^2\xi - \alpha\sigma\left(\frac{\beta}{\delta}\right)\varphi^2\xi + 2\alpha\kappa\left(\frac{\sigma^2}{\delta^2}\right)\varphi^2\varepsilon \\ & - \alpha^2\kappa\left(\frac{\sigma}{\delta^2}\right)\varphi^2\varepsilon - 2\,\alpha\sigma^2\left(\frac{\beta}{\delta^2}\right)\varphi^2\varepsilon + \alpha^2\sigma\left(\frac{\beta}{\delta^2}\right)\varphi^2\varepsilon + 2\alpha\kappa\sigma\beta\left(\frac{\mu}{\delta}\right)\varphi > 0. \end{aligned}$$

From equation (13),

$$\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0 \tag{18}$$

where

$$e_1 = \kappa - \alpha + \sigma - \beta - \gamma + \xi - \rho z^* + \mu x^* - \tau z^* - \delta y^* + \varphi y^* + \varepsilon y^* > 0,$$

$$\begin{aligned} e_2 = {} & \kappa\sigma - \alpha\kappa + \alpha\beta - \sigma\beta + \alpha\gamma - \kappa\gamma - \sigma\gamma + \beta\gamma - \alpha\xi + \kappa\xi + \sigma\xi - \beta\xi + \rho\tau(z^*)^2 - \delta\varphi(y^*)^2 \\ & - \delta\varepsilon(y^*)^2 + \varphi\varepsilon(y^*)^2 - \rho\kappa z^* - \alpha\mu x^* + \kappa\mu x^* + \sigma\mu x^* + \rho\beta z^* - \beta\mu x^* + \rho\gamma z^* \\ & + \alpha\tau z^* - \alpha\varphi y^* - \kappa\delta y^* - \sigma\tau z^* + \sigma\varphi y^* - \rho\xi z^* - \alpha\varepsilon y^* + \kappa\varepsilon y^* + \sigma\varepsilon y^* + \beta\delta y^* \\ & - \beta\varepsilon y^* + \tau\gamma z^* + \gamma\delta y^* - \gamma\varphi y^* - \tau\xi z^* - \delta\xi y^* + \varphi\xi y^* - \rho\varphi y^* z^* - \tau\mu x^* z^* \\ & - \mu\delta x^* y^* + \mu\varphi x^* y^* - \rho\varepsilon y^* z^* + \delta\varphi x^* y^* + \tau\delta y^* z^* > 0, \end{aligned}$$

$$\begin{aligned} e_3 = {} & \alpha\kappa\gamma - \kappa\sigma\gamma - \alpha\beta\gamma + \sigma\beta\gamma - \alpha\kappa\xi + \kappa\sigma\xi + \alpha\beta\xi - \sigma\beta\,\xi - \alpha\kappa\mu x^* + \kappa\sigma\mu x^* + \alpha\,\beta\mu x^* - \sigma\beta\mu x^* \\ & + \rho\kappa\gamma z^* - \rho\beta\gamma z^* - \rho\kappa\xi z^* - \alpha\kappa\varepsilon\, y^* + \kappa\sigma\varepsilon y^* + \rho\beta\xi z^* + \alpha\beta\varepsilon y^* - \alpha\tau\gamma z^* \\ & + \alpha\gamma\varphi y^* + \kappa\gamma\delta y^* - \sigma\beta\varepsilon y^* + \sigma\tau\gamma z^* - \sigma\gamma\varphi y^* - \beta\gamma\delta y^* + \alpha\tau\xi z^* - \alpha\varphi\xi y^* \\ & - \kappa\delta\xi y^* - \sigma\tau\xi z^* + \sigma\varphi\xi y^* + \beta\delta\xi y^* - \rho\tau\gamma(z^*)^2 + \rho\tau\xi(z^*)^2 - \alpha\varphi\varepsilon(y^*)^2 \\ & - \kappa\,\delta\varepsilon(y^*)^2 + \sigma\varphi\varepsilon(y^*)^2 + \beta\delta\varepsilon(y^*)^2 + \gamma\delta\varphi(y^*)^2 - \delta\varphi\xi(y^*)^2 - \delta\varphi\varepsilon(y^*)^3 \\ & + \alpha\tau\mu x^* z^* - \alpha\mu\varphi x^* y^* - \kappa\mu\delta x^* y^* - \sigma\tau\mu x^* z^* + \sigma\mu\varphi x^* y^* - \rho\kappa\varepsilon y^* z^* + \beta\mu\delta x^* y^* \\ & + \rho\beta\varepsilon y^* z^* + \rho\gamma\varphi y^* z^* - \gamma\delta\varphi x^* y^* - \rho\varphi\xi y^* z^* - \tau\gamma\delta y^* z^* + \delta\varphi\xi x^* y^* + \tau\delta\xi y^* z^* \\ & - \mu\delta\varphi x^*(y^*)^2 + \mu\delta\varphi(x^*)^2 y^* - \rho\varphi\varepsilon(y^*)^2 z^* + \delta\varphi\varepsilon x^*(y^*)^2 - \rho\varphi\varepsilon x^* y^* z^* \\ & + 2\tau\mu\delta x^* y^* z^* > 0 \end{aligned}$$

The Hurwitz Matrices for equation (14) are given as

$$H_1 = (a_1), \qquad H_2 = \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & 1 \\ 0 & 0 & a_3 \end{pmatrix} \tag{19}$$

With

$$detH_1 = a_1 > 0, \qquad H_2 = a_1 a_2 > 0, \qquad detH_3 = a_1 a_2 > a_3 \text{ for } a_1 \text{ or } a_2 > a_3 \tag{20}$$

Equation (20) is possible because $a_1 > 0, a_2 > 0$ and $a_3 > 0$.

The Hurwitz Matrices for equation (15) are given as

$$H_1 = (b_1), \qquad H_2 = \begin{pmatrix} b_1 & 1 \\ 0 & b_2 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} b_1 & 1 & 0 \\ b_3 & b_2 & 1 \\ 0 & 0 & b_3 \end{pmatrix} \tag{21}$$

with

$$detH_1 = b_1 > 0, \qquad H_2 = b_1 b_2 > 0, \qquad detH_3 = b_1 b_2 > b_3 \text{ for } b_1 \text{ or } b_2 > b_3 \tag{22}$$

Equation (22) is possible because $b_1 > 0, b_2 > 0$ and $b_3 > 0$.

The Hurwitz Matrices for equation (16) are given as

$$H_1 = (c_1), \qquad H_2 = \begin{pmatrix} c_1 & 1 \\ 0 & c_2 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} c_1 & 1 & 0 \\ c_3 & c_2 & 1 \\ 0 & 0 & c_3 \end{pmatrix} \tag{23}$$

with

$$detH_1 = c_1 > 0, \qquad H_2 = c_1 c_2 > 0, \qquad detH_3 = c_1 c_2 > c_3 \text{ for } c_1 \text{ or } c_2 > c_3 \tag{24}$$

Equation (24) is possible because $c_1 > 0, c_2 > 0$ and $c_3 > 0$.

The Hurwitz Matrices for equation (17) are given as

$$H_1 = (d_1), \qquad H_2 = \begin{pmatrix} d_1 & 1 \\ 0 & d_2 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} d_1 & 1 & 0 \\ d_3 & d_2 & 1 \\ 0 & 0 & d_3 \end{pmatrix} \qquad (25)$$

with

$$detH_1 = d_1 > 0, \qquad H_2 = d_1 d_2 > 0, \qquad detH_3 = d_1 d_2 > d_3 \text{ for } d_1 \text{ or } d_2 > d_3 \qquad (26)$$

Equation (26) is possible because $d_1 > 0, d_2 > 0$ and $d_3 > 0$.

The Hurwitz Matrices for equation (18) are given as

$$H_1 = (e_1), \qquad H_2 = \begin{pmatrix} e_1 & 1 \\ 0 & e_2 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} e_1 & 1 & 0 \\ e_3 & e_2 & 1 \\ 0 & 0 & e_3 \end{pmatrix} \qquad (27)$$

with

$$detH_1 = e_1 > 0, \qquad H_2 = e_1 e_2 > 0, \qquad detH_3 = e_1 e_2 > e_3 \text{ for } e_1 \text{ or } e_2 > e_3 \qquad (28)$$

Equation (28) is possible because $e_1 > 0, e_2 > 0$ and $e_3 > 0$.

Hence the proof. These shows that all the five (5) equilibrium points are locally and asymptotically stable. This result is showing that the model equation (1) – (3) has biological meaning.

3 Results (Numerical Simulation)

The simulated results of the model equations (1) – (3) are presented in Figures 1, 2, 3, 4 and 5 using the parameter values and initial conditions in the Table 2.

Table 2: Parameter values and initial condition

Parameter/ Initial Condition	Figure 1	Figure 2	Figure 3	Figure 4	Figure 5
α	0.002	0.002	0.002	0.002	0.002
δ	0.110	0.110	0.030	0.030	0.030
ρ	0.110	0.030	0.030	0.030	0.090
σ	0.210	0.210	0.210	0.210	0.210
β	3.000	3.000	3.000	3.000	3.000
τ	0.110	0.340	0.340	0.030	0.340
φ	0.400	0.400	0.400	0.010	0.100
κ	0.210	0.210	0.210	0.210	0.210
γ	5.000	5.000	5.000	5.000	5.000
μ	0.330	0.330	0.330	0.330	0.330
ε	0.210	0.210	0.210	0.210	0.210
ξ	0.210	0.210	0.210	0.210	0.210
x_0	10.00	10.00	10.00	10.00	10.00
y_0	15.00	15.00	15.00	15.00	15.00
z_0	20.00	20.00	20.00	20.00	20.00

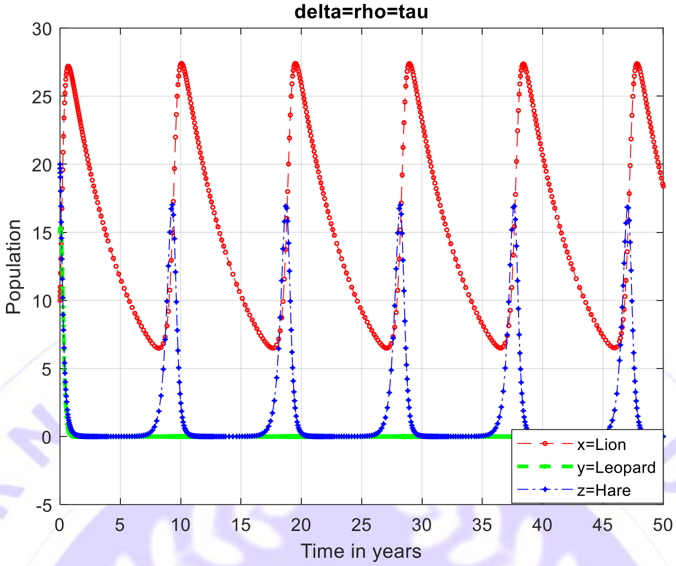

Figure 1: Population dynamic of x, y and z when $\rho = \delta = \tau$

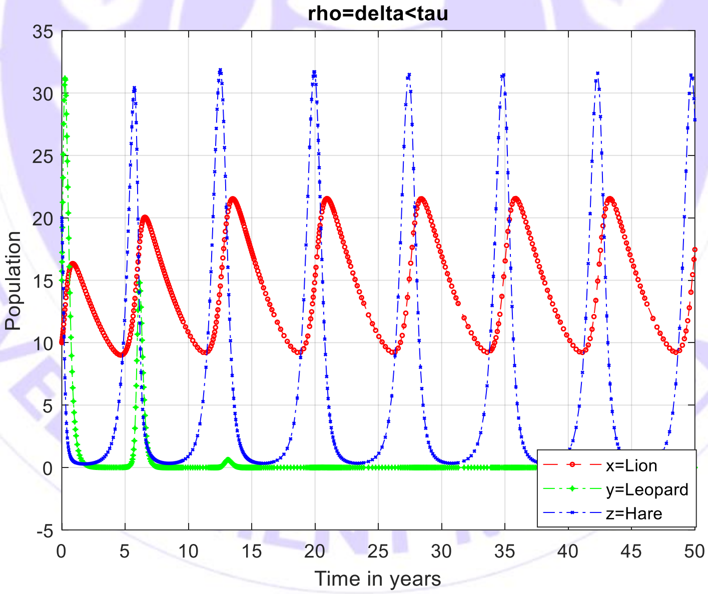

Figure 2: Population dynamic of x, y and z when $\rho = \delta < \tau$

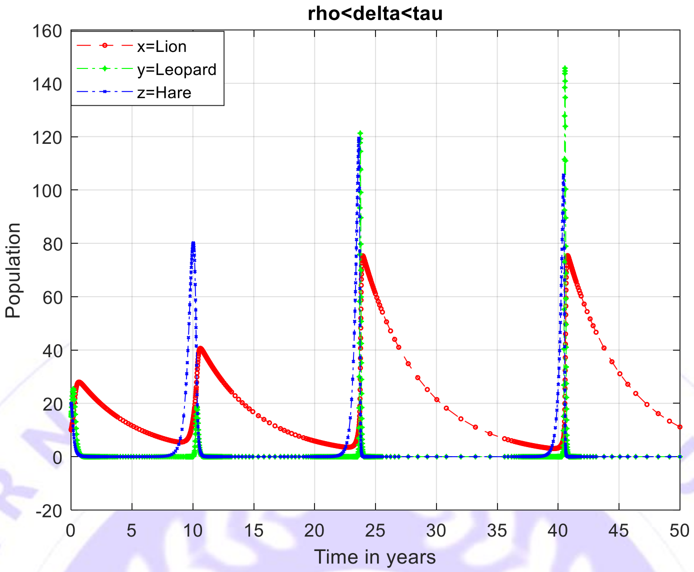

Figure 3: Population dynamic of x, y and z when $\rho < \delta < \tau$

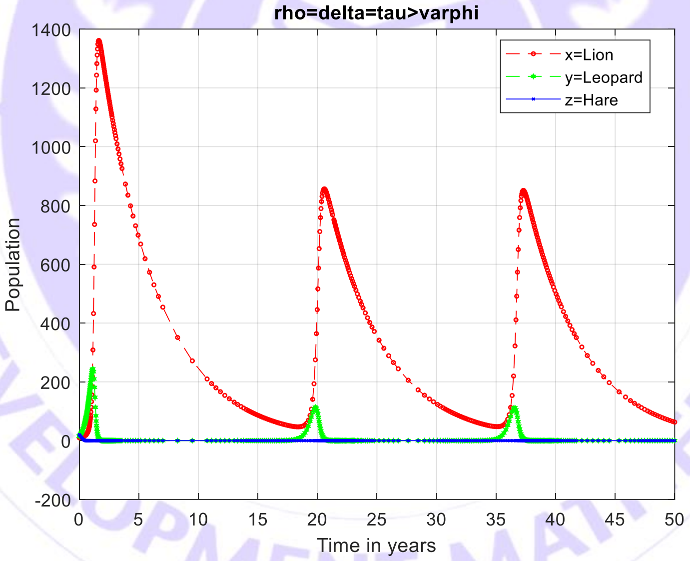

Figure 4: Population dynamic of x, y and z when $\rho = \delta = \tau > \varphi$

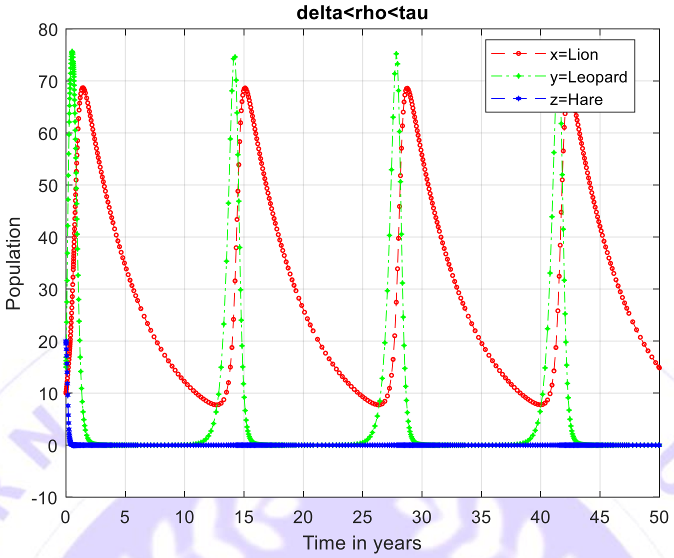

Figure 5: Population dynamic of x, y and z when $\delta < \rho < \tau > \varphi$

4 Discussion

Figure 1 is describing a situation where the population of Lions is equally depended on both the Leopard and Hare populations. The population of lions increases as long as the Leopard or/and Hare population is not zero, but decreases as soon as the population of one or two of the preys is zero and as soon as the Hare population begin to rise, the Lion population will also begin to rise. It can be noticed that the Leopard populations decreases to zero and never rise again which may be attributed to the presence of the lion population and the fact that $\delta = \tau$.

Figure 2 also is describing a situation where the population of Lions is equally depended on both the Leopard and Hare populations but with the Leopard population feed more on Hare population. In this case, the Leopard population stays in the habitat a little longer before going to extinction. But it can still be observed that as Hare population grows/decline, the Lion and Leopard population also grows/decline.

Figure 3 is a situation where the rate at which the Lion population depends on Hare population is less than its rate of depending on the Leopard population and the rate at which the Lion population depends on the Leopard population is less that the rate of Leopard population is depended on the Hare population. It can be notice from the figure that once both Leopard and Hare population increases/decreases, the population of Lion also increases/decreases. Also, because $\delta < \tau$, the population of Leopard is increasing more than that of Hare over time.

Figure 4 shows that the population of Lions is equally depended on both the Leopard and Hare populations. But the rate of Lion feeding on Leopard has no effect on the Leopard population thereby causing the Leopard population to persist (though not many) in the population for a long time to the detriment of the Hares.

Figure 5 is a situation where the rate at which the Lion population depending on Hare population is more than its rate of depending on the Leopard population and the rate at which the Lion population depends on the Leopard population is less that the rate of Leopard population is depending on the Hare population but the rate of Lion population depending on Leopard has not little or no effect on the Leopard population. This will cause the Leopard population to persist for a long time in the population while the population of Hare goes to extinct immediately.

All the results above represent a real scenario.

5 Conclusion

Over time, mathematical models have been utilised to comprehend complex situations in natural sciences and engineering. These mathematical models can either be deterministic or stochastic and are based on ordinary differential equations, partial differential equations and stochastic differential equations. In studying the interrelationships of organisms and their environment, there is need to investigate science of coexistence of two or more species. To this end, it is natural to seek a mathematical formulation of this prey-predator problem and to use it to forecast the behavior of populations of various species at different times.

In order to understand how organisms interact with their surroundings, it is necessary to use mathematical models to explore the science of multispecies cohabitation. There are many instances in nature where one species of animal feeds on another species of animal, which in turn feeds on other species of animals. The first specie is called the first predator, the second is called the second predator/first prey and the third is called the second prey which is the consideration in this paper. This paper studied the interaction between three species of animals (Lion, Leopard and Hare) in an environment using a formulated predator-prey model of system of three nonlinear differential equations of first order with five equilibrium points. The equilibrium points were found using Routh Hurwitz Stability Criterion and found to be locally asymptotically stable. It was also established that, the solution of the model exist and are all positive for any $t \geq 0$. Some numerical simulations were carried out to show the analytical solution of the model and results presented. The results show that the predator-prey presented in this work can be used perfectly to study any species interaction with the same nature with the ones considered in this paper. For further studies, this model can be extended to include four or more species interactions.

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