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A Nonlinear Mathematical Model for the Effect of Diabetes Population on a Community

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ABSTRACT

This work presents a compartmental-based mathematical model of susceptible, diabetes without complications, diabetes with minor and major complications to study the effect of diabetes population on the population dynamics of a community. The model is a system of four nonlinear differential equations of first order. The solution of the model was found to exist and is positive by positivity analysis using a contradiction method. The diabetes-free and the diabetes-endemic equilibrium points were also found to be locally asymptotically stable using the Routh-Hurwitz Stability Criterion for a degree n-polynomial. The numerical simulation of the model was carried out using various scenarios and the results show that diabetes population in a community has great effect on the population dynamics of the community either positively or negatively. The results here represent a real-life scenario thereby making the proposed model realistic.

1. Introduction

Excessive blood sugar causes Diabetes Mellitus (DM), a metabolic disease known as the "silent killer" in medical terminology (World Health Organization, 2023). Diabetes is a historical term from Greek that means "syphon." The word originally meant the act of passing water, or urine, via a syphon. The Latin term "mellitus" means "sweetened" or "like honey." When combined, the word "Diabetes Mellitus" literally referred to a medical condition where a person consistently passed sweetened urine (Kwaghkor and Luga, 2016). Polyurea (frequent urination), polydipsia (feeling thirstier), and polyphagia (feeling hungrier) are a few of the symptoms of diabetes that people may encounter. Many methods exist for diagnosing diabetes, but the most often used one is the glucose tolerance test (GTT), which measures the patient's blood glucose during a fast (World Health Organization, 1994). The three main types of diabetes are (1) Type 1 diabetes, which develops when your pancreas is unable to produce insulin, (2) type 2 diabetes, which develops when your pancreas doesn't produce enough insulin or when your body isn't using the insulin well and (3) gestational diabetes, which develops during pregnancy and typically goes away after the baby is delivered (Kwaghkor and Luga, 2016; Yadav & Maya, 2020; Kwaghkor *et al.*, 2022). A bad lifestyle leads a susceptible individual to become a diabetic. Diabetes is primarily caused by unhealthy lifestyles, which include inactivity, inconsistent eating patterns, and other bad behaviours (Widyaningsih *et al.*, 2018).

Individuals with diabetes who have difficulties are typically already aware of how fragile their physical state is; in contrast, diabetics who do not experience complications have not been able to continue their lifestyle, particularly those who are undiagnosed. Diabetics without difficulties typically lead unhealthy lifestyles in day-to-day activities. Lifestyle "transmission" can occur when healthy people connect with diabetics who maintain unhealthy lifestyles. The effect of "transmission" of a lifestyle is occurrence. Diabetic prevalence rises with incidence (Hill *et al.*, 2013; Widyaningsih *et al.*, 2018; American Diabetes Association, 2018, 2020).

In the field of natural sciences and engineering, mathematical models have been used for understanding complicated phenomena throughout time (Alkali *et al.*, 2023; Kwaghkor *et al.*, 2018; Kwaghkor *et al.*, 2019; Kwaghkor *et al.*, 2021; Orapine *et al.*, 2023. Numerous studies have been conducted on the dynamics of diabetes. Diabetes Complication (DC) Model has provided a mathematical model to explain the prevalence of diabetics (Widyaningsih *et al.*, 2018). The DC model was first introduced by Boutayeb *et al.* (2004) to find out how many changes of diabetics without complications (D) and diabetics with complications (C). The DC model was developed into a susceptible diabetes complication (SDC) model (Hill *et al.*, 2013). As contained in Widyaningsih *et al.* (2018), the specific set of susceptible people led to this extension. In addition to the existing works, Modu *et*

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al. (2021) extended the DC Model by dividing the complication compartment into minor complications (C_1) and major complications (C_2) in order to monitor the dynamics of populations with minor and major complications. In this study, we shall extend the work by Modu *et al.* (2021) to include the susceptible population which will enable us study the effect of diabetes population on the population dynamics of a community. The rest of this work is sectionalized as follows: methods presented in section 2, results in section 3, discussion in section 4 and conclusion in section 5.

2. Methods

2.1 The model

We obtained the nonlinear system of ordinary differential equation of the model from the Schematic diagram of the model in figure 1 below.

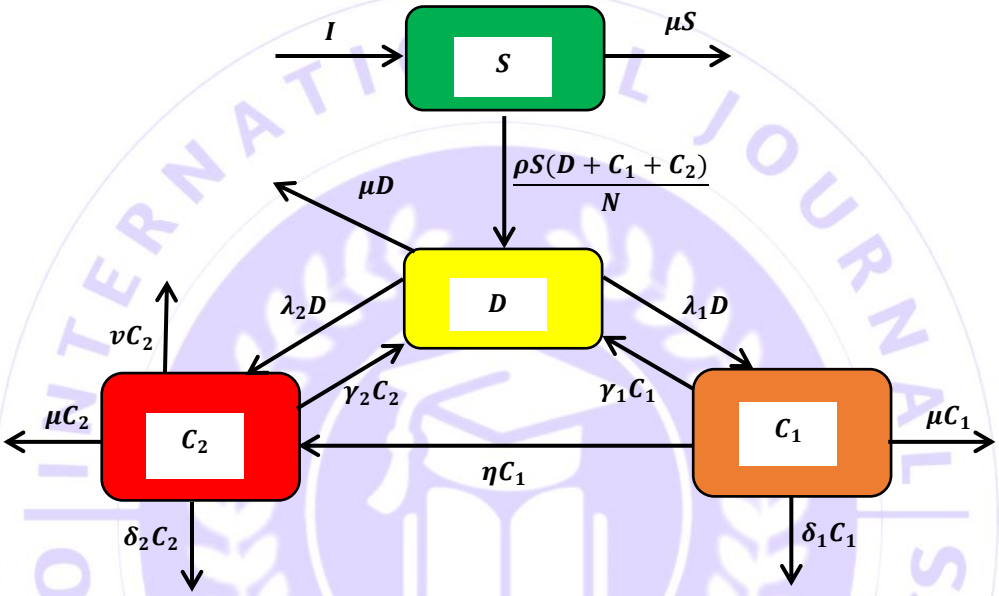

Figure 1: The Schematic diagram of the model

From Figure 1, $S(t)$ is the susceptible class (Number of people without diabetes), $D(t)$ is the number of diabetic patients without complications, $C_1(t)$ is the number of diabetic patients with minor complications, $C_2(t)$ is the number of diabetic patients with major complications, I is the number persons becoming susceptible which is defined as $I = \alpha N$, ρ is the diabetic recruitment rate, μ is the natural death rate, δ_1 and δ_2 are rates of death caused by minor and major complications respectively, λ_1 and λ_2 are rates of developing minor and major complications respectively, γ_1 and γ_2 are rates of recovery from minor and major complications respectively, η is the rate of developing major complications from minor complications and v is the rate of which diabetic patient with major complications become severely disabled.

2.2 The model equations

Based on the flow diagram, the model equations are presented as follows

$$\frac{dS}{dt} = I - \frac{\rho S(D+C_1+C_2)}{N} - \mu S \tag{1}$$

$$\frac{dD}{dt} = \frac{\rho S(D+C_1+C_2)}{N} - (\lambda_1 + \lambda_2 + \mu)D + \gamma_1 C_1 + \gamma_2 C_2 \tag{2}$$

$$\frac{dC_1}{dt} = \lambda_1 D - (\gamma_1 + \eta + \delta_1 + \mu)C_1 \tag{3}$$

$$\frac{dC_2}{dt} = \eta C_1 + \lambda_2 D - (\gamma_2 + \mu + \delta_2 + v)C_2 \tag{4}$$

where $N = S + D + C_1 + C_2$.

The model equations (1) – (4) can also be presented as

$$\frac{ds}{dt} = I - \frac{\rho S(D+C_1+C_2)}{N} - \mu S \tag{5}$$

$$\frac{dD}{dt} = \frac{\rho S(D+C_1+C_2)}{N} - AD + \gamma_1 C_1 + \gamma_2 C_2 \tag{6}$$

$$\frac{dC_1}{dt} = \lambda_1 D - BC_1 \tag{7}$$

$$\frac{dC_2}{dt} = \eta C_1 + \lambda_2 D - EC_2 \tag{8}$$

$$where\ A = \lambda_1 + \lambda_2 + \mu, B = \gamma_1 + \eta + \delta_1 + \mu\ and, E = \gamma_2 + \mu + \delta_2 + v$$

2.3 Model Analyses

2.3.1 Positivity and boundedness of solutions

Theorem 1: The region $\varphi = \left\{(S(t), D(t), C_1(t), C_2(t)) \in \mathbb{R}_+^4 : S(t), D(t), C_1(t), C_2(t) \geq 0, N(t) \leq \frac{I}{\mu}\right\}$ is positively invariant with respect to the model (5) – (8). Then, the solutions $S(t), D(t), C_1(t), C_2(t)$ are all positive for $t \geq 0$.

Proof.

We show by the method of contradiction in Bhunu *et al.* (2009) that φ is positively invariant. Consider the following cases

Case 1: If there exist $t_1 > 0$ such that, $S(t_1) = 0, S'(t_1) < 0, D(t) > 0, C_1(t) > 0, C_2(t) > 0$ for $0 < t < t_1$, or

Case 2: If there exist $t_2 > 0$ such that, $D(t_2) = 0, D'(t_2) < 0, S(t) > 0, C_1(t) > 0, C_2(t) > 0$ for $0 < t < t_2$, or

Case 3: If there exist $t_3 > 0$ such that, $C_1(t_3) = 0, {C_1}'(t_3) < 0, S(t) > 0, D(t) > 0, C_2(t) > 0$ for $0 < t < t_3$, or

Case 4: If there exist $t_4 > 0$ such that, $C_2(t_4) = 0, {C_2}'(t_4) < 0, S(t) > 0, D(t) > 0, C_1(t) > 0$ for $0 < t < t_4$.

From Case 1, $\frac{dS}{dt} = I > 0$. This is a contradiction which means that $S(t)$ remains positive.

From Case 2, $\frac{dD}{dt} = \frac{\rho S(C_1+C_2)}{N} + \gamma_1 C_1 + \gamma_2 C_2 > 0$. This is a contradiction which means that $D(t)$ remains positive.

From Case 3, $\frac{dC_1}{dt} = \lambda_1 D > 0$. This is a contradiction which means that $C_1(t)$ remains positive.

From Case 4, $\frac{dC_2}{dt} = \mu C_1 + \lambda_2 D > 0$. This is a contradiction which means that $C_2(t)$ remains positive.

So in all the cases $S(t), D(t), C_1(t),$ and $C_2(t)$ remain positive. Since $N(t) \geq S(t) + D(t) + C_1(t) + C_2(t)$, it implies that $\alpha - \mu N(t) - \delta_1 C_1 - \delta_2 C_2 - VC_2 \leq N'(t) \leq \alpha - \mu N(t)$. This result implies that $N(t)$ is bounded and that all the solutions $S(t), D(t), C_1(t),$ and C_2(t) in φ stays in it. Hence the proof.

2.3.2 Equilibrium points of the model

The Model (5) – (8) has two equilibrium Points: The Diabetes Free Equilibrium (DFE) point and the Diabetes Endemic Equilibrium (DEE) point.

i. The Diabetes Free Equilibrium (DFE)

At the **DFE** equilibrium points, $\frac{ds}{dt} = \frac{dD}{dt} = \frac{dC_1}{dt} = \frac{dC_2}{dt} = D = C_1 = C_2 = 0$ *which when substituting in* (5) – (8) and manipulating yields to

$$\boldsymbol{DFE} = (S^*, D^*, {C_1}^*, {C_2}^*) = \left(\frac{I}{\mu}, 0, 0, 0\right) \tag{9}$$

ii. The Diabetes Endemic Equilibrium (DEE)

At the **DEE** equilibrium points, $\frac{ds}{dt} = \frac{dD}{dt} = \frac{dC_1}{dt} = \frac{dC_2}{dt} = 0$ *which when solving for* S, D, C_1 *and* C_2 *in* (5) – (8) yields to

$$\boldsymbol{DEE} = (S^{**}, D^{**}, {C_1}^{**}, {C_2}^{**}) \tag{10}$$

where

$$S^{**} = \frac{ABE - B\lambda_2\gamma_2 + \eta\lambda_1\gamma_2 + \lambda_1\gamma_1 E}{B\rho E - B\rho\lambda_2 + \eta\rho\lambda_1 + \rho\lambda_1 E},$$

$$D^{**} = \left(\frac{B}{\rho}\right)\left(\frac{E}{(ABE + B\lambda^2\gamma_2 - \eta\lambda^1\gamma_2 - \lambda^1\gamma_1 E)(BE - B\lambda^2 + \eta\lambda^1 + \lambda^1 E)}\right)(BI\rho\lambda_2 - BI\rho E - I\eta\rho\lambda_1 - I\rho\lambda_1 E + ABN\mu E - BN\mu\lambda_2\gamma_2 + N\mu\eta\lambda_1\gamma_2 + N\mu\lambda_1\gamma_1 E),$$

$${C_1}^{**} = \left(\frac{1}{\rho}\right)\lambda_1\left(\frac{E}{(ABE + B\lambda_1\gamma_2 - \eta\lambda_1\gamma_2 - \lambda_1\gamma_1 E)(BE - B\lambda_2 + \eta\lambda_1 + \lambda_1 E)}\right)(BI\rho\lambda_2 - BI\rho E - I\eta\rho\lambda_1 - I\rho\lambda_1 E + ABN\mu E - BN\mu\lambda_2\gamma_2 + N\mu\eta\lambda_1\gamma_2 + N\mu\lambda_1\gamma_1 E), \text{ and}$$

$${C_2}^{**} = \left(\frac{1}{\rho}\right)\left(\frac{\eta\lambda_1 - B\lambda_2}{(ABE + B\lambda_2\gamma_2 - \eta\lambda_1\gamma_2 - \lambda_1\gamma_1 E)(BE - B\lambda_2 + \eta\lambda_1 + \lambda_1 E)}\right)(BI\rho\lambda_2 - BI\rho E - I\eta\rho\lambda_1 - I\rho\lambda_1 E + ABN\mu E - BN\mu\lambda_2\gamma_2 + N\mu\eta\lambda_1\gamma_2 + N\mu\lambda_1\gamma_1 E).$$

The Diabetes Endemic Equilibrium (DEE) can only have biological meaning if

$$(BI\rho\lambda_2 - BI\rho E - I\eta\rho\lambda_1 - I\rho\lambda_1 E + ABN\mu E - BN\mu\lambda_2\gamma_2 + N\mu\eta\lambda_1\gamma_2 + N\mu\lambda_1\gamma_1 E) > 0 \tag{11}$$

Equation (11) when manipulated gives

$$\frac{BI\rho\lambda_2 + ABN\mu E + N\mu\eta\lambda_1\gamma_2 + N\mu\lambda_1\gamma_1 E}{BI\rho E + I\eta\rho\lambda_1 + I\rho\lambda_1 E + BN\mu\lambda_2\gamma_2} > 1 \tag{12}$$

Equation (12) is the Diabetes Endemic condition, R_0 (Basic Reproduction Value). By replacing I with αN in (12) and manipulating gives

$$R_0 = \frac{B\alpha\rho\lambda_2 + AB\mu E + \mu\eta\lambda_1\gamma_2 + \mu\lambda_1\gamma_1 E}{B\alpha\rho E + \alpha\eta\rho\lambda_1 + \alpha\rho\lambda_1 E + B\mu\lambda_2\gamma_2} \tag{13}$$

The basic reproductive value R_0 is typically defined by Diekmann e*t al.* (1990) as: the average number of secondary infections produced by an infected individual during his/her entire life as infectious (infectious period) when introduced in a population of susceptible. On computing, if $R_0 \leq 1$, it shows that the disease will die out with time. While $R_0 > 1$ shows that the disease will persist in the population for a longer period of time (Trawicki, 2017).

iii. Local Stabilities of the DFE and DEE

Theorem 2: Routh-Hurwitz Stability Criterion

Consider degree n polynomial $p(s) = s^{n} + a_1 s^{n-1} + \ldots + a_{\{n-1\}} s + a_{\{n\}} s^{0}$ and Table 1:

Table 1: The Routh Array of the given n −polynomial

s^{n}	1	a_2	a_4	a_6	...
s^{n-1}	a_1	a_3	a_5	a_7	...
s^{n-2}	b_1	b_2	b_3	...	
s^{n-3}	c_1	c_2	...		
:					
s^{1}	*	*			
s^{0}	*				

Assume that the necessary condition for stability holds, if $a_1, a_2, \ldots, a_{\{n\}} > 0$, then, $p(s)$ is stable if and only if all entries in the first column of Table 1 are positive.

Proof.

The proof of Theorem 2 is done in two parts: Part 1 and Part 2.

Part 1: DFE

The linearized version of the nonlinear system (1) – (4) by taken the Jacobian is presented as

$$J_{DFE} = \begin{pmatrix} -\mu & -\frac{\rho I}{\mu} & -\frac{\rho I}{\mu} & -\frac{\rho I}{\mu} \\ \mu & \frac{\rho I}{\mu} - A & \frac{\rho I}{\mu} + \gamma_1 & \frac{\rho I}{\mu} + \gamma_2 \\ 0 & \lambda_1 & -B & 0 \\ 0 & \lambda_2 & \eta & E \end{pmatrix} \text{ with the characteristic equation}$$

$$F_1 X^4 + F_2 X^3 + F_3 X^2 + F_4 X + F_5 = 0 \tag{14}$$

where

$F_1 = 1$

$F_2 = \left(B - A + \mu - E - \left(\frac{I}{\mu}\right)\rho\right),$

$F_3 = \left(E\left(A - B - \mu + \left(\frac{I}{\mu}\right)\rho\right) - \mu\left(A + \left(\frac{I}{\mu}\right)\rho\right) - \lambda_1\left(\gamma_1 + \left(\frac{I}{\mu}\right)\rho\right) - \lambda_2\left(\gamma_2 + \left(\frac{I}{\mu}\right)\rho\right) - B\left(A - \mu + \left(\frac{I}{\mu}\right)\rho\right)\right),$

$$\begin{aligned} F_4 = \Bigg(E\Bigg(\mu\left(A + \left(\frac{I}{\mu}\right)\rho\right) + \lambda_1\left(\gamma_1 + \left(\frac{I}{\mu}\right)\rho\right) + B\left(A - \mu + \left(\frac{I}{\mu}\right)\rho\right)\Bigg) - B\mu\left(A + \left(\frac{I}{\mu}\right)\rho\right) \\ - \lambda_1\left(A + \left(\frac{I}{\mu}\right)\rho\right)\left(\gamma_1 + \left(\frac{I}{\mu}\right)\rho\right) - \lambda_2\left(A + \left(\frac{I}{\mu}\right)\rho\right)\left(\gamma_2 + \left(\frac{I}{\mu}\right)\rho\right) \\ + \lambda_2\left(\gamma_2 + \left(\frac{I}{\mu}\right)\rho\right)\left(A - B - \mu + \left(\frac{I}{\mu}\right)\rho\right) - \eta\lambda_1\left(\gamma_2 + \left(\frac{I}{\mu}\right)\rho\right) \\ + \lambda_1\left(\gamma_1 + \left(\frac{I}{\mu}\right)\rho\right)\left(A - \mu + \left(\frac{I}{\mu}\right)\rho\right)\Bigg) \text{ and} \end{aligned}$$

$$F_5 = \left(E\left(\mu\left(A + \left(\frac{I}{\mu}\right)\rho\right) + \lambda_1\left(\gamma_1 + \left(\frac{I}{\mu}\right)\rho\right) + B\left(A - \mu + \left(\frac{I}{\mu}\right)\rho\right)\right) - B\mu\left(A + \left(\frac{I}{\mu}\right)\rho\right) \right.$$
$$- \lambda_1\left(A + \left(\frac{I}{\mu}\right)\rho\right)\left(\gamma_1 + \left(\frac{I}{\mu}\right)\rho\right) - \lambda_2\left(A + \left(\frac{I}{\mu}\right)\rho\right)\left(\gamma_1 + \left(\frac{I}{\mu}\right)\rho\right)$$
$$+ \lambda_2\left(\gamma_2 + \left(\frac{I}{\mu}\right)\rho\right)\left(A - B - \mu + \left(\frac{I}{\mu}\right)\rho\right) - \eta\lambda_1\left(\gamma_2 + \left(\frac{I}{\mu}\right)\rho\right)$$
$$\left. + \lambda_1\left(\gamma_1 + \left(\frac{I}{\mu}\right)\rho\right)\left(A - \mu + \left(\frac{I}{\mu}\right)\rho\right)\right).$$

The necessary condition for stability holds since $F_1, F_2, F_3, F_4, F_5 > 0$. The Routh table for (9) is presented below

Table 2: Routh table for equation (9)

X^4	1	F_3	F_5
X^3	F_2	F_4	0
X^2	$b_1 = \frac{F_2F_3 - F_4}{F_3}$	$b_2 = \frac{F_2F_5}{F_3}$	0
X^1	$c_1 = \frac{b_1F_4 - b_2F_2}{b_1}$	0	0
X^0	F_5	0	0

From Table 2, the entries of the first column are all positive if and only if $F_2F_3 - F_4 > 0$ and $b_1F_4 - b_2F_2 > 0$. Hence, the DFE is locally stable for $F_2F_3 - F_4 > 0$ and $b_1F_4 - b_2F_2 > 0$.

Part 2: DEE

The linearized version of the nonlinear system (1) – (4) by taken the Jacobian is presented as

$$J_{DFE} = \begin{pmatrix} -\frac{\rho(D^{**}+C_1{}^{**}+C_2{}^{**})}{N} - \mu & -\frac{\rho S^{**}}{N} & -\frac{\rho S^{**}}{N} & -\frac{\rho S^{**}}{N} \\ \frac{\rho(D^{**}+C_1{}^{**}+C_2{}^{**})}{N} + \mu & \frac{\rho S^{**}}{N} - A & \frac{\rho S^{**}}{N} + \gamma_1 & \frac{\rho S^{**}}{N} + \gamma_2 \\ 0 & \lambda_1 & -B & 0 \\ 0 & \lambda_2 & \eta & E \end{pmatrix}$$ with the characteristic equation

$$G_1X^4 + G_2X^3 + G_3X^2 + G_4X + G_5 = 0 \quad (15)$$

where

$$G_1 = 1$$

$$G_2 = \left(A + B + \mu - E - \left(\frac{1}{N}\right)S^{**}\rho + \left(\frac{1}{N}\right)\rho(D^{**} + C_1{}^{**} + C_2{}^{**})\right)$$

$$G_3 = \left(B\left(A + \mu - \left(\frac{1}{N}\right) S^{**}\rho + \left(\frac{1}{N}\right)\rho(D^{**} + {C_1}^{**} + {C_2}^{**}) \right) \right.$$
$$- E\left(A + B + \mu - \left(\frac{1}{N}\right) S^{**}\rho + \left(\frac{1}{N}\right)\rho(D^{**} + {C_1}^{**} + {C_2}^{**}) \right)$$
$$+ \left(A - \left(\frac{1}{N}\right) S^{**}\rho \right)\left(\mu + \left(\frac{1}{N}\right)\rho(D^{**} + {C_1}^{**} + {C_2}^{**}) \right) - \lambda_1\left(\gamma_1 + \left(\frac{1}{N}\right) S^{**}\rho \right)$$
$$\left. - \lambda_2\left(\gamma_2 + \left(\frac{1}{N}\right) S^{**}\rho \right) + \left(\frac{1}{N^2}\right) S^{**}\rho^2(D^{**} + {C_1}^{**} + {C_2}^{**}) \right)$$

$$G_4 = \left(\lambda_1\left(\left(A - \left(\frac{1}{N}\right) S^{**}\rho \right)\left(\gamma_1 + \left(\frac{1}{N}\right) S^{**}\rho \right) + \left(\frac{1}{N^2}\right) S^{**}\rho^2(D^{**} + {C_1}^{**} + {C_2}^{**}) \right) \right.$$
$$+ \lambda_2\left(\left(A - \left(\frac{1}{N}\right) S^{**}\rho \right)\left(\gamma_2 + \left(\frac{1}{N}\right) S^{**}\rho \right) + \left(\frac{1}{N^2}\right) S^{**}\rho^2(D^{**} + {C_1}^{**} + {C_2}^{**}) \right)$$
$$+ B\left(\left(A - \left(\frac{1}{N}\right) S^{**}\rho \right)\left(\mu + \left(\frac{1}{N}\right)\rho(D^{**} + {C_1}^{**} + {C_2}^{**}) \right) + \left(\frac{1}{N^2}\right) S^{**}\rho^2(D^{**} + {C_1}^{**} + {C_2}^{**}) \right)$$
$$- E\left(B\left(A + \mu - \left(\frac{1}{N}\right) S^{**}\rho + \left(\frac{1}{N}\right)\rho(D^{**} + {C_1}^{**} + {C_2}^{**}) \right) \right.$$
$$+ \left(A - \left(\frac{1}{N}\right) S^{**}\rho \right)\left(\mu + \left(\frac{1}{N}\right)\rho(D^{**} + {C_1}^{**} + {C_2}^{**}) \right) - \lambda_1\left(\gamma_1 + \left(\frac{1}{N}\right) S^{**}\rho \right)$$
$$\left. + \left(\frac{1}{N^2}\right) S^{**}\rho^2(D^{**} + {C_1}^{**} + {C_2}^{**}) \right)$$
$$- \lambda_1\left(\gamma_1 + \left(\frac{1}{N}\right) S\rho \right)\left(A + \mu - \left(\frac{1}{N}\right) S^{**}\rho + \left(\frac{1}{N}\right)\rho(D^{**} + {C_1}^{**} + {C_2}^{**}) \right)$$
$$- \lambda_2\left(\gamma_2 + \left(\frac{1}{N}\right) S^{**}\rho \right)\left(A + B + \mu - \left(\frac{1}{N}\right) S^{**}\rho + \left(\frac{1}{N}\right)\rho(D^{**} + {C_1}^{**} + {C_2}^{**}) \right)$$
$$\left. - \eta\lambda_1\left(\gamma_2 + \left(\frac{1}{N}\right) S^{**}\rho \right) \right)$$

$$
\begin{aligned}
G_5 = \Bigg(& \eta\Bigg(\lambda_1\Bigg(\Big(A-\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\Big(\gamma_2+\Big(\frac{1}{N}\Big)S^{**}\rho\Big)+\Big(\frac{1}{N^2}\Big)S^{**}\rho^2(D^{**}+{C_1}^{**}+{C_2}^{**})\Bigg)\\
& +B\lambda_1\Big(\gamma_2+\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\Bigg)\\
& -E\Bigg(\lambda_1\Bigg(\Big(A-\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\Big(\gamma_1+\Big(\frac{1}{N}\Big)S^{**}\rho\Big)+\Big(\frac{1}{N^2}\Big)S^{**}\rho^2(D^{**}+{C_1}^{**}+{C_2}^{**})\Bigg)\\
& +B\Bigg(\Big(A-\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\Big(\mu+\Big(\frac{1}{N}\Big)\rho(D^{**}+{C_1}^{**}+{C_2}^{**})\Big)+\Big(\frac{1}{N^2}\Big)S^{**}\rho^2(D^{**}+{C_1}^{**}+{C_2}^{**})\Bigg)\\
& -\lambda_1\Big(\gamma_1+\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\Big(A+\mu-\Big(\frac{1}{N}\Big)S^{**}\rho+\Big(\frac{1}{N}\Big)\rho(D^{**}+{C_1}^{**}+{C_2}^{**})\Big)\Bigg)\\
& -\lambda_2\Bigg(\Big(A-\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\Bigg(\Big(A-\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\Big(\gamma_2+\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\\
& +\Big(\frac{1}{N^2}\Big)S^{**}\rho^2(D^{**}+{C_1}^{**}+{C_2}^{**})\Bigg)+\lambda_1\Big(\gamma_1+\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\Big(\gamma_2+\Big(\frac{1}{N}\Big)S\rho\Big)\\
& -\Big(\frac{1}{N}\Big)\rho\Bigg(\Big(\frac{1}{N}\Big)S^{**}\rho\Big(\gamma_2+\Big(\frac{1}{N}\Big)S^{**}\rho\Big)-\Big(\frac{1}{N}\Big)S^{**}\rho\Big(\mu+\Big(\frac{1}{N}\Big)\rho(D^{**}+{C_1}^{**}+{C_2}^{**})\Big)\Bigg)(D^{**}\\
& +{C_1}^{**}+{C_2}^{**})\Bigg)\\
& +\Bigg(\lambda_2\Bigg(\Big(A-\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\Big(\gamma_2+\Big(\frac{1}{N}\Big)S^{**}\rho\Big)+\Big(\frac{1}{N^2}\Big)S^{**}\rho^2(D^{**}+{C_1}^{**}+{C_2}^{**})\Bigg)\\
& -\eta\lambda_1\Big(\gamma_2+\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\Bigg)\Big(A+B+\mu-\Big(\frac{1}{N}\Big)S^{**}\rho+\Big(\frac{1}{N}\Big)\rho(D^{**}+{C_1}^{**}+{C_2}^{**})\Big)\\
& -\lambda_2\Big(\gamma_2+\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\Bigg(B\Big(A+\mu-\Big(\frac{1}{N}\Big)S^{**}\rho+\Big(\frac{1}{N}\Big)\rho(D^{**}+{C_1}^{**}+{C_2}^{**})\Big)\\
& +\Big(A-\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\Big(\mu+\Big(\frac{1}{N}\Big)\rho(D^{**}+{C_1}^{**}+{C_2}^{**})\Big)-\lambda_1\Big(\gamma_1+\Big(\frac{1}{N}\Big)S^{**}\rho\Big)\\
& +\Big(\frac{1}{N^2}\Big)S^{**}\rho^2(D^{**}+{C_1}^{**}+{C_2}^{**})\Bigg)\Bigg)
\end{aligned}
$$

The necessary condition for stability holds since $G_1, G_2, G_3, G_4, G_5 > 0$. The Routh table for (10) is presented below

Table 3: Routh table for equation (10)

X^4	1	G_3	G_5
X^3	G_2	G_4	0
X^2	$b_1 = \frac{G_2G_3 - G_4}{G_3}$	$b_2 = \frac{G_2G_5}{G_3}$	0
X^1	$c_1 = \frac{b_1G_4 - b_2G_2}{b_1}$	0	0
X^0	G_5	0	0

From Table 3, the entries of the first column are all positive if and only if $G_2G_3 - G_4 > 0$ and $b_1G_4 - b_2G_2 > 0$. Hence, the DFE is locally stable for $G_2G_3 - G_4 > 0$ and $b_1G_4 - b_2G_2 > 0$.

3. Numerical Simulation

The numerical results are presented here by the aid of MATLAB using the variable/parameter values in Table 4 below

Table 4: Variable and Parameter Values, (Expt. = Experiment)

Variable/ Parameter	**Expt. 1**	**Expt. 2**	**Expt. 3**	**Expt. 4**	**Expt. 5**	**Expt. 6**	**Source**
μ	0.0143	0.0143	0.0143	0.0143	0.0143	0.0143	Huo and Qiu (2014)
λ_1	0	0	0.33	0	0.33	0.33	Assumed
λ_2	0	0	0.40	0.40	0	0.40	Assumed
γ_1	0.042	0.042	0	0	0.042	0.042	Boutetyab, *et al*., (2004)
γ_2	0.038	0.038	0	0.038	0	0.038	Boutetyab, *et al*., (2004)
δ_1	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	IDF (2014)
δ_2	0.005	0.005	0.005	0.005	0.005	0.005	IDF (2014)
v	0.05	0.05	0.05	0.05	0.05	0.05	Boutetyab, *et al*., (2004)
ρ	0	0.2213	0.2213	0.2213	0.2213	0.00001	Assumed
α	0.0143	0.0143	0.0143	0.0143	0.0143	0.0143	Assumed
η	0	0	0.03	0.03	0.03	0.03	Modu et al., (2021)
S(0)	6000	6000	6000	6000	6000	6000	Assumed
D(0)	5000	5000	5000	5000	5000	5000	Assumed
$C_1(0)$	3000	3000	3000	3000	3000	3000	Assumed

$C_2(0)$	1000	1000	1000	1000	1000	1000	Assumed
N(0)	15000	15000	15000	15000	15000	15000	Assumed

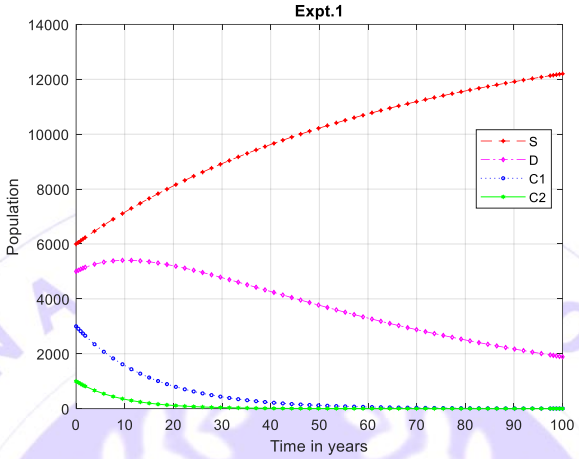

Figure 2: Population dynamics with $\rho = \eta = \lambda_1 = \lambda_2 = 0$

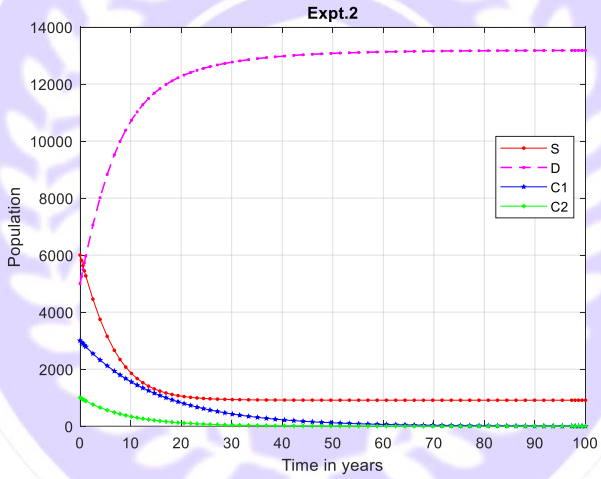

Figure 3: Population dynamics with $\rho = 0.2213,\ \eta = \lambda_1 = \lambda_2 = 0$

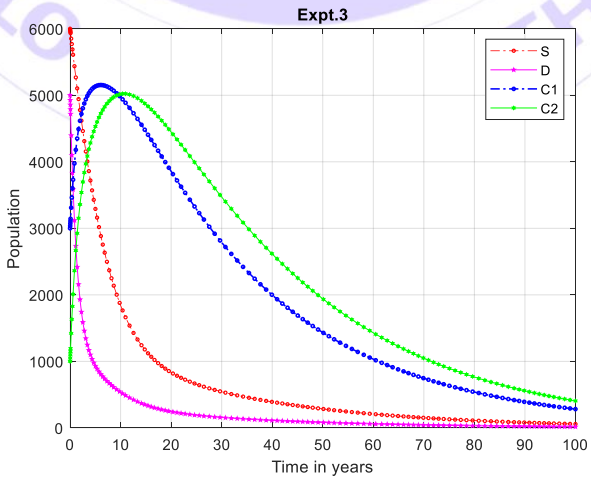

Figure 4: Population dynamics with $\gamma_1 = \gamma_2 = 0$

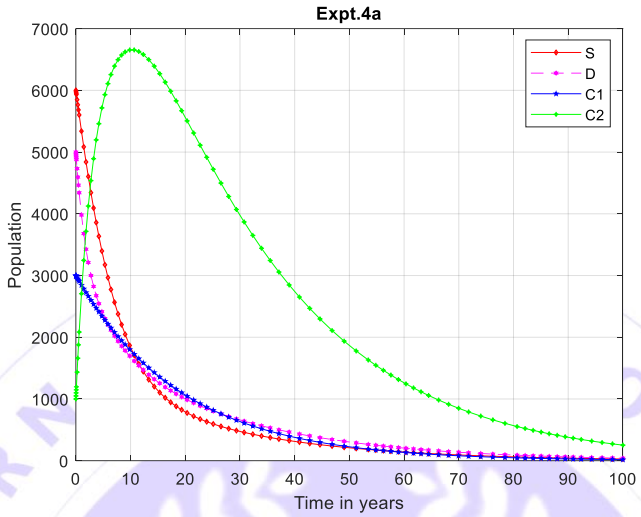

Figure 5a: Population dynamics with $\gamma_1 = \lambda_1 = 0$ and $\gamma_2 < \lambda_2$

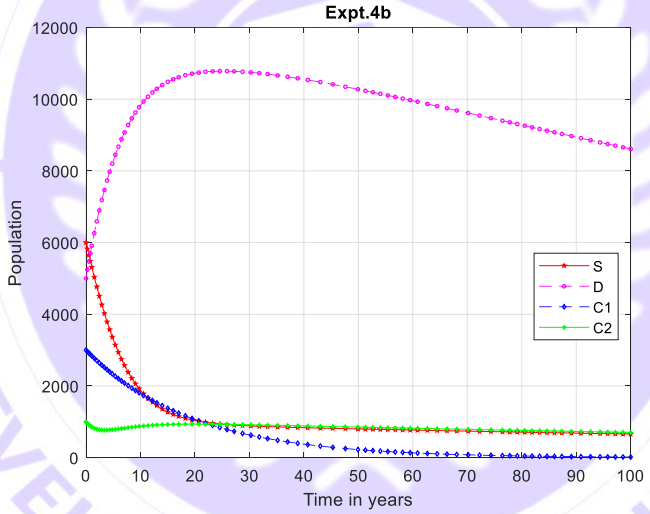

Figure 5b: Population dynamics with $\gamma_1 = \lambda_1 = 0$ and $\gamma_2 > \lambda_2$

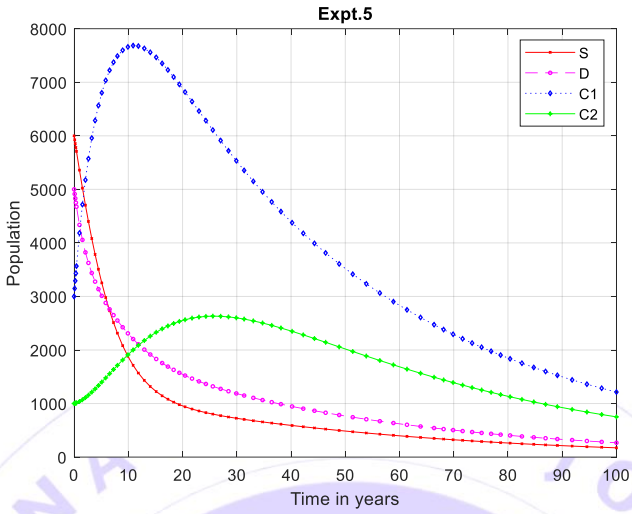

Figure 6: Diabetes population dynamics with $\gamma_2 = \lambda_2 = 0$

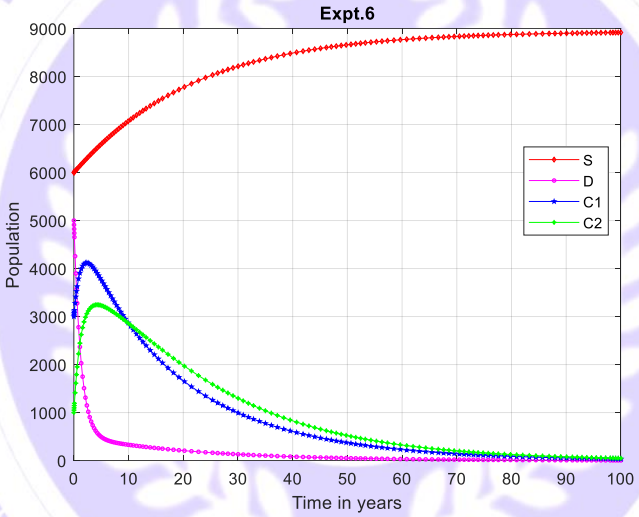

Figure 7: Diabetes population dynamics with reduced $\rho = 0.00001$

4. Discussion

Figure 2 is describing a situation where the rate at which the susceptible population are becoming diabetic is zero (that is, the recruitment rate is zero) indicating that there is no rate of becoming diabetic without complications, with minor and major complications. It can be seen from the figure that if there are no persons entering the diabetes without complications compartment (which could be as a result of a good lifestyle in the society), diabetes with minor and major complications compartments will die off over time. The result is showing that the susceptible class will keep growing unhindered. This conforms to reality.

Figure 3 is describing a situation where the rates of having minor and major complications are zero, thereby causing the population of diabetics without complications grow unrestricted. This indicates that there is a possibility of people living with diabetes for the rest of their lives without getting into complications from it. This also conforms to reality.

Figure 4 is describing a situation where there is no recovery either from minor and major complications. This shows that people will die more in that population causing the entire population in the community to decline over time. This also conforms to reality.

Figure 5a is a situation where people are not entering nor coming out of the minor complication compartment through the diabetes without complication compartment thereby causing a rise in the major complication compartment over time. While Figure 5b is a similar case to that of Figure 5a, except that in Figure 5b, $\gamma_2 > \lambda_2$ thereby causing a rise in the compartment of diabetes without complications.

Figure 6 is a situation where people are not entering nor coming out of the major complications compartment through the diabetes without complication compartment thereby causing a rise in the minor complication compartment over time.

Figure 7 is a situation where the rate of entering into the compartment of diabetes without complications is reduced to the lowest level thereby reducing the diabetes population within the community with time.

5. Conclusion

This work presents a compartmental-based mathematical model of susceptible, diabetes without complications, diabetes with minor and major complications to study the effect of diabetes population on the population dynamics of a community. The model is a system of four nonlinear differential equations of first order. The solution of the model was found to exist and is positive by positivity analysis using a contradiction method. The diabetes-free and the diabetes-endemic equilibrium points are found to be locally asymptotically stable using the Routh-Hurwitz Stability Criterion for a degree n-polynomial. The numerical simulation of the model was carried out using various scenarios and the results show that diabetes population in a community has great effect on the population dynamics of the community. The results here represent a physical life scenario thereby making the proposed model realistic. The results show that if there are no persons entering the diabetes without complications compartment (which could be as a result of a good lifestyle in the society), diabetes with minor and major complications compartments will die off over time. The result is showing that the susceptible class will keep growing unhindered. One of the results also indicates that there is a possibility of people living with diabetes for the rest of their lives without getting into complications from it. The results also show that as there is no recovery either from minor and major complications, that people will die more in that population causing the entire population in the community to decline over time. When people are not entering nor coming out of the minor complication compartment through the diabetes without complication compartment thereby causing a rise in the major complication compartment over time. The opposite of this result is that $\gamma_2 > \lambda_2$ thereby causing a rise in the compartment of diabetes without complications. Also, there is a situation where people are not entering nor coming out of the major complications compartment through the diabetes without complication compartment thereby causing a rise in the minor complication compartment over time. Also, there is a situation where the rate of entering into the compartment of diabetes without complications is reduced to the lowest level thereby reducing the diabetes population within the community with time.

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