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Expressing Partial Order-Preserving Transformations as Products of Nilpotents

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ABSTRACT

Let S be a semigroup with zero, then an element a∈S is called nilpotent, if there exists a positive integer n such that $a^n = 0$. In the partial transformation semigroup on X, where X is a non-empty set, denoted by P_X, the empty map is the zero of P_X. Further, nilpotency is a structural property of fundamental importance which arises in many situations, for instance in linear algebra, we have nilpotent matrices and much more in other branches of mathematics. In this study, we will investigate, some elements of semigroups of partial one-to-one order-preserving transformations (denoted by IO_n) and partial order-preserving transformations (denoted by PO_n), and some basic properties of nilpotent transformations. We have studied certain properties of nilpotent elements in $\mathcal{J}_X$, symmetric inverse semigroup, where X is finite and infinite, certain properties of nilpotent elements in P_X, descriptions of the elements of ⟨N⟩, where N is the set of nilpotents of IO_n and we used the same letter N to denote that of PO_n, and used them to establish the following results, in this paper (that is, we have used the knowledge that we have acquired to establish the following results): - a partial ono-to-one order-preserving transformation α, has from 1 to $n-3$ fixed, $n \geq 4$ and there is an upper jump of length two(2) between $n-3$ and n, and lower jumps of length one(1) and one(1), immediately after $n-3$, can be expressed as a product of three nilpotents. - a partial one-to-one order-preserving transformation α, has from 1 to $n-4$ fixed, $n \geq 5$ and there is an upper jump of length three (3) between $n-4$ and n and lower jumps of length two (2) and one (1), immediately after $n-4$, can be expressed as a product of fewer than three nilpotents. - a partial order-preserving transformation α, has from 1 to $n-4$ fixed, but $n-3$ belongs to A_{n-4}, $n \geq 5$ and there is an upper jump of length two(2) between A_{n-4} and A_n, and lower jumps of length two(2) and one(1), immediately after $n-4$, cannot be expressed as a product of fewer than three nilpotents. - a partial order-preserving transformation α, has from 1 to $n-5$ fixed, but $n-4$ belongs to A_{n-5}, $n \geq 6$ and there is an upper jump of length three (3) between A_{n-5} and A_n, and lower jumps of length three (3) and one (1), immediately after n-5, can be expressed as a product of fewer than three nilpotents. - a transformation $\alpha \in PT_n$, partial transformation semigroup, is not nilpotent if and only if the graph Γ_α of the transformation α has a loop, and - a transformation $\alpha \in IS_n$, partial one-to-one transformation semigroup, which has a nilpotent path and a permutation part is not nilpotent.

1. Introduction

Let S be a semigroup with zero, then an element $a \in S$ is called nilpotent, if there exists a positive integer n such that $a^n = 0$. In the partial transformation semigroup on X, where X is a non-empty set, denoted by P_X, the empty map is the zero of P_X.

Further, nilpotency is a structural property of fundamental importance which arises in many situations, for instance in linear algebra, we have nilpotent matrices and much more in other branches of mathematics.

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Howie (1996) characterized the elements of T_X, the full transformation semigroup on X, where X is a non-empty set, with composition of maps as binary operation, which can be written as a product of idempotents in T_X. These are precisely the singular self-maps on X, denoted by $sing_X$.

Now in this study we consider P_X, the partial transformation semigroup. Since P_X contains a zero, it contains nilpotents, and so it is natural to ask for a description of the sub-semigroup of P_X generated by the nilpotents of P_X. This is part of Sullivan (1987) Semigroup Generated by Nilpotent Transformations of P_X. Apart from the Subsemigroup generated by nilpotents, we do have; Semigroups of Partial One-to-One Order-Preserving Mappings (denoted by IO_n) and Partial Order-Preserving Transformations (denoted by PO_n). The semigroups PO_n and IO_n are regular and inverse semigroups, respectively. There are papers Howie and Marques-Smith (1984) and Gomes and Howie (1987), on Inverse Semigroups Generated by Nilpotent Transformations and Nilpotents in Finite Symmetric Inverse Semigroups, deal with investigating certain properties of nilpotent elements in symmetric inverse semigroup on X, where X is infinite set and investigating some properties of strictly partial transformations related to nilpotent elements, where the set is finite, respectively. In a paper Garba (1991), on Nilpotents in Semigroups of Partial One-to-One Order-Preserving mappings: he has studied, descriptions of elements of $\langle N \rangle$, where N is a set of nilpotents of IO_n. Garba extended the idea to PO_n, in a paper Garba (1994), Nilpotents in Semigroups of Partial Order-Preserving Transformations. And also, there is a paper, Ali *et al.* (2018) on "On the Ranks of Certain Semigroups of Order-Preserving Partial Isometries of Finite Chain. In this paper, they study the nilpotent in ODP_n and investigate the ranks of two subsemigroups of ODP_n; the nilpotent generated subsemigroup $\langle N \rangle$ and the subsemigroup $L(n,r) = \{\alpha \in ODP_n : |im(\alpha)| \leq r\}$. Here, in this study, we will investigate some elements of PO_n and IO_n and some basic properties of nilpotent transformations.

Note that: Section 1.3 of this paper contains statements with proofs of the results that we have established. The binary operation used in Section 1.3 is the usually composition of mappings. For the reader, firstly, Study the preliminaries. IO_n is a subsemigroup of PO_n.

2. Preliminaries

Definition 2.1: Let M be a finite set, say $M = \{m_1, m_2, \ldots, m_n\}$ where n is a nonnegative integer. A transformation of M is an array of the following form;

$$\alpha = \begin{pmatrix} m_1 & m_2 & . & . & . & m_n \\ k_1 & k_2 & . & . & . & k_n \end{pmatrix} \tag{1}$$

where all $k \in M$. If $x \in M$, say $x = m$, the element k will be called the value of the transformation α at the element x and will be denoted by $\alpha(x)$. Apart from the transformations of M, we shall also, consider partial transformation of M, that is, transformations of the form $\alpha : A \to M$, where $A = \{l_1, l_2, \ldots, l_k\}$ is a subset of M. Note that the set A can be empty. Again, the element α can be written in the following tabular form:

$$\alpha = \begin{pmatrix} l_1 & l_2 & . & . & . & l_k \\ \alpha(l_1) & \alpha(l_2) & . & . & . & \alpha(l_k) \end{pmatrix} \tag{2}$$

We may also write $\alpha : M \to M$ for a partial transformation, having in mind that such α is only defined on some elements from M (Ganyushkin & Mazorchuk, 2009).

Definition 2.2: Let S be a nonempty set, and $*$ be a binary operation on S. Then $(S,*)$ is called a semigroup provided that * is associative, that is, $a * (b * c) = (a * b) * c \; for \; all \; a, b, c \in S$ Ganyushkin and Mazorchuk (2009).

Definition 2.3: A partial one-to-one order-preserving transformation semigroup, is denoted by IO_n. An element α of IO_n, is called nilpotent if $\alpha^k = 0$ for some $k \geq 1$ (Garba, 1991).

Definition 2.4: A transformation of the following form

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix} \tag{3}$$

is called partial one-to-one order-preserving, where $a_1 < a_2 < \ . \ . \ . \ < a_r$, $b_1 < b_2 < \ . \ . \ . \ < b_r$ and $r < n$ (Garba, 1991).

Definition 2.5: A transformation of the following form

$$\alpha = \begin{pmatrix} A_1 & A_2 & . & . & . & A_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix} \tag{4}$$

where for each $a_i \in A_i,\ a_i < a_{i+1}\ \ (i = 1,\ .\ .\ .\ , r)$ and $b_1 < b_2 < \ .\ .\ . < b_r$. Let $x_i = min\{x : x \in A_i\}$ and $y_i = max\{x : x \in A_i\}$. For $i = 1,\ .\ .\ .\ r$, let $S_i = \{x \in X_n :\ x_i \le x \le y_i\}$, and for $i = 1,\ .\ .\ .\ , r-1$, $T_i = \{x \in X_n :\ y_i < x < x_{i+1}\}$. Let $T_0 = \{x \in X_n : x < x_1\}$ and $T_r = \{x \in X_n : x > y_r\}$.
is called partial order-preserving (Garba, 1994).

Definition 2.6: A graph is an ordered triple $G = (V(G), E(G), I_G)$, where $V(G)$ is a nonempty set, E(G) is a set disjoint from $V(G)$, and I_G is an incidence relation that associates with each element of $E(G)$ an unordered pair of elements (same or disjoint) of $V(G)$. Elements of $V(G)$ are called the vertices (or nodes or points) of G, and elements of $E(G)$ are called the edges (or lines) of G. $V(G)$ *and* $E(G)$ are the vertex set and edge set of G, respectively (Balarishnan & Ranganathan, 2012).

Definition 2.7: A directed graph D is an ordered triple $(V(D), A(D), I_D)$, where $V(D)$ is a nonempty set called the set of vertices of D; $A(D)$ is a set disjoint from $V(D)$, called the set of arcs of D; and I_D is an incidence map that associates with each arc of D, an ordered pair of vertices of D. If a is an arc of D, and $I_D(a) = (u, v), u$ is called the tail of a, and v is the head of a. The arc a is said to join v with u, u and v are called the ends of a. A directed graph is also called a digraph (Balarishnan & Ranganathan, 2012).

Some information about partial one-to-one order-preserving transformations, can be found in (Garba, 1994).
We shall say that an element α in IO_n, has height r or that $h(\alpha) = r$ if and only if $|dom\alpha| = |im\alpha| = r$. If we let $\alpha, \beta \in IO_n$ be of height r, and suppose that

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix}\ ,\ \ \beta = \begin{pmatrix} c_1 & c_2 & . & . & . & c_r \\ d_1 & d_2 & . & . & . & d_r \end{pmatrix}\ , \tag{5}$$

where $a_1 < a_2 < \cdot\ \cdot\ \cdot < a_r$, $\ b_1 < b_2 < \cdot\ \cdot\ \cdot < b_r$, $\ c_1 < c_2 < \cdot\ \cdot\ \cdot < c_r$, $\ d_1 < d_2 < \cdot\ \cdot\ \cdot < d_r$, then we have $im\alpha = im\gamma$, $dom\beta = dom\gamma$, where

$$\gamma = \begin{pmatrix} c_1 & c_2 & . & . & . & c_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix} \tag{6}$$

is clearly an element in IO_n.

The semigroup IO_n has $n + 1$ $\ \mathcal{J} - classes\ \ J_0, J_1, .\ .\ .\ , J_n$ where $J_r\ \ (r = 1\ ,\ .\ .\ .\ , n)$ consists of all elements of height r. Each $\mathcal{H}$-class consists of only one element. Thus

$$|J_r| = \binom{n}{r}^2 \quad \text{and}$$

$$|IO_n| = \sum_{r=0}^{n} \binom{n}{r}^2 \tag{7}$$

An element α of IO_n is called nilpotent if $\alpha^k = 0$ for some $k \ge 1$.

Lemma 2.1: Let $\alpha \in J_r$, $r < n$. Then α is nilpotent if and only if $x\alpha \ne x$ for every $x \in dom\alpha$ (Garba, 1994).

Proof: If $\alpha = 0$ (the empty map) the result is trivial. We therefore suppose that $dom\alpha \neq \emptyset$. It is clear that if $x\alpha = x$ for some $x \in dom\alpha$, then α cannot be nilpotent. For we will have $x = x\alpha = x\alpha^2 = . \ . \ .$

Conversely, suppose that $x\alpha \neq x$ for every $x \in dom(\alpha)$. Then $im(\alpha) \neq dom(\alpha)$. For if $im(\alpha) = dom(\alpha)$, then the order-preserving property implies that $x\alpha = x$ for every $x \in dom(\alpha)$. Now $(dom(\alpha))\alpha = im(\alpha) \neq dom(\alpha)$ and so $dom(\alpha^2) \subseteq dom(\alpha)$ (properly). We now show that for $k = 2,3, \ . \ . \ .$

$$dom(\alpha^k) \neq \emptyset \text{ implies } dom(\alpha^{k+1}) \subset dom(\alpha^k) \text{ (properly).} \tag{8}$$

For suppose by way of contradiction that $dom(\alpha^{k+1}) = dom(\alpha^k) \neq \emptyset$ for some $k \geq 2$. Then

$$dom(\alpha^k) = dom(\alpha \cdot \alpha^k) = \big(im(\alpha) \cap dom(\alpha^k)\big)\alpha^{-1}. \tag{9}$$

Thus $|im(\alpha) \cap dom(\alpha^k)| = |dom(\alpha^k|$ and so, since the sets are finite

$$im(\alpha) \cap dom(\alpha^k) = dom(\alpha^k). \tag{10}$$

From (9) and (10) it follows that $im(\alpha^k) = \big(dom(\alpha^k)\big)\alpha = dom(\alpha^k)$. But this will mean that $x\alpha^k = x$ for every $x \in dom(\alpha^k)$. We also have $im(\alpha^{k+1}) \subseteq im(\alpha^k) = dom(\alpha^k) = dom(\alpha^{k+1})$. Since α is one-to-one, so also is α^{k+1} and hence $im(\alpha^{k+1}) = dom(\alpha^{k+1})$. Again, this will mean that $x\alpha^{k+1} = x$ for every $x \in dom(\alpha^{k+1})$. Now $x = x\alpha^{k+1} = x\alpha^k \cdot \alpha = x\alpha$ for every $x \in dom(\alpha^{k+1}) \subseteq dom(\alpha)$ contrary to hypothesis. We thus have a strict descent

$$dom(\alpha) \supset dom(\alpha^2) \supset . \ . \ . \tag{11}$$

and hence there exist $m \geq 1$ such that $dom(\alpha^m) = \emptyset$, that is, such that $\alpha^m = 0$.

Let

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix} \tag{12}$$

be an element in IO_n, and suppose that

$$\alpha = n_1 n_2 \ . \ . \ . \ n_k \ , \tag{13}$$

a product of k nilpotents. Each n_i must be of height at least r. If any n_i is of height greater than r, we can replace it by one of the heights exactly r simply by removing the redundant elements. Accordingly, we may assume that in the product (13) each n_i has height precisely r. Thus $dom(\alpha) = dom(n_1)$. If

$$a_i n_1 = c_i \ \ (i = 1,2, \ . \ . \ . \ , r) \tag{14}$$

we must have $c_i \neq a_i$ for all i. Hence in expressing α as product of nilpotents, we shall seek a set $\{c_1, \ c_2, \ \ . \ . \ . \ , c_r\}$ for which $c_i < c_{i+1}$. Such a set may or may not exist. For example, if $n = 5$ and $dom(\alpha) = \{1, \ 3, \ 5\}$, then no such set exists. (one can verify this by looking at all the 10 subsets of cardinality 3.)

We shall say that α has an upper jump of length k [a lower jump of length k] if there exists an i such that

$$a_{i+1} = a_i + k + 1 \ \ [b_{i+1} = b_i + k + 1] \, . \tag{15}$$

If $a_1 = k + 1$ [$b_1 = k + 1$] and $k \geq 1$, we shall also say that α has an upper jump of length k [a lower jump of length k]. For example,

$$\alpha = \begin{pmatrix} 1 & 3 & 6 & 7 \\ 2 & 5 & 6 & 8 \end{pmatrix} \tag{16}$$

has upper jumps of length 1 and 2, lower jumps of length 1, 2, and 1. We shall refer to the sum of the lengths of upper jumps of α as the total upper jump of α, and denote it by $j^*(\alpha)$. Likewise, the sum of lengths of lower jumps of α is the total lower jump of α, and we denote it by $j_*(\alpha)$. In the example above, $j^*(\alpha) = 3$ and $j_*(\alpha) = 4$. It is easy to verify that if α is given by (12), then

Lemma 2.2: $j^*(\alpha) = a_r - r$, $j_*(\alpha) = b_r - r$ (Garba, 1994).

Theorem 2.1: For $n \geq 2$. Let α defined by

$$a_i\alpha = b_i \text{ for } 1 \leq i \leq r \tag{17}$$

be an element of j_r, where $r < n$. Then α is not a product of nilpotents if and only if α satisfies one of the following:

(i) $a_1 = 1, a_r = n$ and all upper jumps are of length 1 ,

(ii) $b_1 = 1$, $b_r = n$ and all lower jumps are of length 1 (Garba, 1994).

The Proof of Theorem 2.1 can be found in (Garba, 1994).

A Short note on Decomposing of Charts: charts are elements of finite symmetric inverse semigroups, denoted by "C_n" , $n = 1,2,3, \ . \ . \ .$. The multiplication is function composition. A chart $\alpha \in C_n$ if and only if $\alpha : \boldsymbol{d}\alpha \to \boldsymbol{r}\alpha$ is a one-to-one function whose domain $\boldsymbol{d}\alpha$ and range $\boldsymbol{r}\alpha$ are subsets of $N = \{1,2, \ . \ . \ . \ , n\}$. Since permutations of N are charts in C_n, the symmetric group S_n (of all permutations of N) is a subgroup of C_n.

Having S_n as a subgroup of C_n, we might suspect that the disjoint cycle decomposition of permutations somehow extends to charts, i.e., given any chart $\alpha \in C_n$, we desire to "decompose" $\alpha = \alpha_1 \ \cdots \ \alpha_k$ into certain "atomic charts" $\alpha_1, \ . \ . \ . \ , \alpha_k$ (Stephen, 1996).

Definition 2.8: A semigroup S is an inverse semigroup if every element of S has a unique inverse, that is to say, the system of equations $a = axa, x = xax$ has a unique solution for x (Petrich, 1996).

Definition 2.9: A sub-semigroup of S which is a group with respect to the multiplication inherited from S will be called a subgroup, where S is a semigroup (Howie, 1995).

Definition 2.10: The order of a is defined as the order of the sub-semigroup $\langle a \rangle$, where $\langle a \rangle$ is the monogenic sub-semigroup of S generated by the element a, where S is a semigroup (Howie, 1976).

3. Methodology

We have studied certain properties of nilpotent elements in $\mathcal{J}(X)$, symmetric inverse semigroup, where X is finite and infinite, certain properties of nilpotent elements in P_X and descriptions of the elements of $\langle N \rangle$, where N is the set of nilpotents of IO_n and we used the same letter N to denote that of PO_n, and used them to establish the results (that is, we have used the knowledge that we have acquired to establish the results). where $\langle N \rangle = N \cup N^2 \cup \cdots \cup N^k$, but $\langle N \rangle \neq N \cup N^2 \cup \cdots \ N^{k-1}$, k is positive integer.

a. Descriptions of Some Elements of PO_n and IO_n in term of Nilpotents

Corollary 3.1: A partial one-to-one order-preserving transformation, say α,

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & . & . & . & n-3 & n \\ 1 & 2 & 3 & . & . & . & n-3 & n-1 \end{pmatrix} \tag{18}$$

can be expressed as a product of three nilpotents.

proof: Firstly, study the preliminaries. To express the transformation α as product of three nilpotent transformations, we will create image set and domains which will give us the three nilpotent transformations.

The following are the possible image set and domains:

$\{2,\ 3,\ 4,\ .\ .\ .\ .\ ,n-2,\ \ n-1\}$ for image set

$\{2,\ 3,\ 4,\ .\ .\ .\ .\ ,n-2,\ \ n\}$ for domain of the transformation

and $\{3,\ 4,\ 5,\ .\ .\ .\ ,n-1\ \ ,n\}$ for domain of the transformation

Now, consider the sets $\{2,\ 3,\ 4,\ .\ .\ .\ ,n-2\ \ ,n-1\}$ and $\{3,\ 4,\ 5,\ .\ .\ .\ .\ ,n-1\ ,n\}$, we have

$$\begin{pmatrix} 1 & 2 & 3 & . & . & . & n-3 & n \\ 1 & 2 & 3 & . & . & . & n-3 & n-1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & . & . & . & n-3 & n \\ 2 & 3 & 4 & . & . & . & n-2 & n-1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 & . & . & . & n-2 & n-1 \\ 3 & 4 & 5 & . & . & . & n-1 & n \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 & 5 & . & . & . & n-1 & n \\ 1 & 2 & 3 & . & . & . & n-3 & n-1 \end{pmatrix} \qquad (19)$$

Lemma 3.1: A partial one-to-one order-preserving transformation, say α', is given by

$$\alpha' = \begin{pmatrix} 1 & 2 & 3 & . & . & . & n-4 & n \\ 1 & 2 & 3 & . & . & . & n-4 & n-1 \end{pmatrix}, \qquad (20)$$

can be written as a product of fewer than three nilpotents, for $n \geq 5$.

Proof: The following are the possible image sets for the nilpotent transformations
,

$\{2,\ 3,\ 4,\ .\ .\ .\ .\ ,\ n-3,\ \ n-2\}$,

$\{2,\ 3,\ 4,\ .\ .\ .\ .\ ,\ n-3,\ \ n-1\}$, and (21)

$\{3,\ 4,.\ .\ .\ .\ ,\ n-2,\ \ n-1\}$

and the possible domains for the nilpotent transformations are as follows:

$\{2,\ 3,\ 4,\ .\ .\ .\ .\ ,\ n-3,\ \ n-2\}$,

$\{2,\ 3,\ 4,\ .\ .\ .\ .\ ,\ n-3,\ \ n\}$ and (22)

$\{3,\ 4,\ .\ .\ .\ .\ ,n-2,\ \ n\}$.

Using the first ones of the image sets and domains of the nilpotent transformations, we have the required expression

$$\alpha' = \begin{pmatrix} 1 & 2 & 3 & . & . & . & n-4 & n \\ 2 & 3 & 4 & . & . & . & n-3 & n-2 \end{pmatrix}\begin{pmatrix} 2 & 3 & 4 & . & . & . & n-3 & n-2 \\ 1 & 2 & 3 & . & . & . & n-4 & n-1 \end{pmatrix} \qquad (23)$$

Corollary 3.1: Determine the total upper jump in the first row,

$$\alpha = \begin{pmatrix} A_1 & A_2 & . & . & . & A_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix}$$

where for each $a_i \in A_i$, $a_i < a_{i+1}$ $(i = 1, 2, \ . \ . \ . \ , r)$ and $b_1 < b_2 < \ . \ . \ . \ < b_r$. Let $x_i = min\{x: x \in A_i\}$ and $y_i = max\{x: x \in A_i\}$. For $i = 1, 2, \ . \ . \ . \ , r$, let $S_i = \{x \in X_n: x_i \leq x \leq y_i\}$, and for $r = 1, 2, \ . \ . \ . \ , r-1$, $T_i = \{x \in X_n: y_i < x < x_{i+1}\}$. Let $T_0 = \{x \in X_n: x < x_1\}$ and $T_r = \{x \in X_n: x > y_r\}$.

Proof: Let $|T_0|$, $|T_r|$, $|T_i|$ and $|S_i \setminus A_i|$ be the number of elements in T_0, T_r, T_i and $S_i \ \setminus A_i$ respectively. Therefore, total gap in the first row is given by

$$j^*(\alpha) = |T_0| + |T_r| + \textstyle\sum_{i=1}^{r} |S_i \setminus A_i| + \sum_{i=1}^{r-1} |T_i| \tag{24}$$

$S_i \setminus \ A_i$ means the set S_i without the set A_i

If there are no jumps in the $A_i s$, $i = 1, 2, \ . \ . \ . \ , r$, we have

$$j^*(\alpha) = |T_0| + |T_r| + \textstyle\sum_{i=1}^{r-1} |T_i| \tag{25}$$

Lemma 3.2: An element α in PO_n is given by

$$\alpha = \begin{pmatrix} A_1 & A_2 & . & . & . & A_{n-4} & A_n \\ 1 & 2 & . & . & . & n-4 & n-1 \end{pmatrix} \text{ and}$$

cannot be expressed as a product of fewer than three nilpotents, $n \geq 5$ and $n-3 \in A_{n-4}$.

Proof: From the above PO_n, $|T_{n-4}| \geq 2$ and max of A_{n-4} is $n-3$.

$1 \in A_1, |T_{n-4}| \geq 2.$

Construct the following image set of the nilpotent transformations

$\{3, 4, \ . \ . \ . \ , n-2 \ \ , n-1\}$ and the domains of the nilpotent transformation

$$\{2, 3, \ . \ . \ . \ , n-3 \ \ , n-2\}, \{2, 3, \ . \ . \ . \ , n-3 \ \ , n\}. \tag{26}$$

$\{3, 4, \ . \ . \ . \ , n-2, \ n\}$ and $\{3, 4, \ . \ . \ . \ , n-2, \ n-1\}$,

one of the domains coincides with the image set, but it cannot provide nilpotent transformation together with the image set of α, therefore the partial order-preserving transformation α, cannot be expressed as a product of fewer than three nilpotents.

Further can be expressed as a product of three nilpotents.

$$\begin{pmatrix} A_1 & A_2 & . & . & . & A_{n-4} & A_n \\ 1 & 2 & . & . & . & n-4 & n-1 \end{pmatrix} =$$

$$\begin{pmatrix} A_1 & A_2 & . & . & . & A_{n-4} & A_n \\ 3 & 4 & . & . & . & n-2 & n-1 \end{pmatrix} \begin{pmatrix} 3 & 4 & . & . & . & n-2 & n-1 \\ 2 & 3 & . & . & . & n-3 & n-2 \end{pmatrix} \begin{pmatrix} 2 & 3 & . & . & . & n-3 & n-2 \\ 1 & 2 & . & . & . & n-4 & n-1 \end{pmatrix}$$

Lemma 3.3: An element α in PO_n is given by

$$\alpha = \begin{pmatrix} A_1 & A_2 & . & . & . & A_{n-5} & A_n \\ 1 & 2 & . & . & . & n-5 & n-1 \end{pmatrix}, \tag{27}$$

can be expressed as a product of fewer than three nilpotents, $n \geq 6$ and $n-4 \in A_{n-5}$.

Proof: From the above PO_n, $1 \in A_1, |T_{n-5}| \geq 2, n \geq 6$ *and max* of A_{n-5} is $n-4$.

$1 \in A_1, |T_{n-5}| \geq 2$

To create the nilpotent transformations,
consider the image sets $\{3, \ 4, \ . \ . \ . \ . \ , n-3, \ n-1\}$ and $\{3, \ 4, \ . \ . \ . \ , n-3, \ n-2\}$

and for the domains, we have $\{2, \ 3, \ . \ . \ . \ . \ , n-4, \ n-3\}$,

$\{2, \ 3, \ . \ . \ . \ , n-4, \ n-2\}$, $\{2, \ 3, \ . \ . \ . \ . \ , \ n-4, \ n\}$,

$\{3, \ 4, \ . \ . \ . \ , n-3, \ n-2\}$ and $\{3, \ 4, \ . \ . \ . \ n-3, \ n\}$.

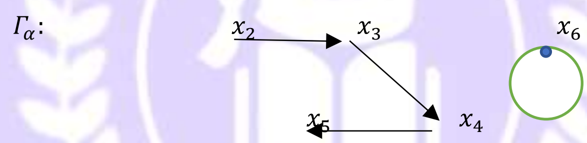
Therefore

$$\alpha = \begin{pmatrix} A_1 & A_2 & . \ . \ . & A_{n-5} & A_n \\ 3 & 4 & . \ . \ . & n-3 & n-2 \end{pmatrix} \begin{pmatrix} 3 & 4 & . \ . \ . & n-3 & n-2 \\ 1 & 2 & . \ . \ . & n-5 & n-1 \end{pmatrix} \tag{28}$$

Corollary 3.2: The transformation $\alpha \in PT_n$, where PT_n is a partial transformation semigroup, is not nilpotent if and only if the graph Γ_α of the transformation α has a loop.

Proof: Suppose that $\alpha \in PT_n$ is nilpotent. Then no element $x \in dom(\alpha)$ such that $x\alpha = x$. Hence, the graph Γ_α of the transformation $\alpha \in PT_n$ has no loop.

Conversely, suppose that the graph Γ_α of the transformation $\alpha \in PT_n$ has a loop. Then it has the following form:



(29)

and the transformation $\alpha \in PT_n$ of the above graph, has the following form:

$$\alpha = \begin{pmatrix} x_2 & x_3 & x_4 & x_6 \\ x_3 & x_4 & x_5 & x_6 \end{pmatrix} \tag{30}$$

By applying the proof of the lemma that says: α is nilpotent if and only if $x\alpha \neq x \ \forall \ x \in dom(\alpha)$, thus the transformation $\alpha \in PT_n$ is not nilpotent.

Corollary 3.3: Partial one-to-one order-preserving transformation α, is defined by

$$a_i\alpha = b_i, i = 1,2, \ . \ . \ . \ . \ , r \ , r < n.$$

That is,

$$\alpha = \begin{pmatrix} a_1 & a_2 & . \ . \ . & a_r \\ b_1 & b_2 & . \ . \ . & b_r \end{pmatrix}, \tag{31}$$

can be expressed as product of nilpotents, if one of the following conditions is satisfied:

(i) $a_1 \neq 1, \ b_1 \neq 1, \ a_r = n, \ b_r = n$

(ii) $a_1 \neq 1, \ b_1 \neq 1, \ a_r \neq n, \ b_r = n$

(iii) $a_1 \neq 1,\ b_1 \neq 1,\ a_r = n,\ b_r \neq n$

(iv) $a_1 \neq 1,\ b_1 \neq 1,\ a_r \neq n,\ b_r \neq n$

Proof: Suppose that $a_1 \neq 1$, $b_1 \neq 1$, $a_r = n$, $b_r = n$ with an upper jump of length one and at least one of length greater than or equal to one and a lower jump of length one and at least one lower jump of length greater than or equal to one.

The partial one-one order-preserving transformation is given by

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix}, \text{ can be an idempotent.} \tag{32}$$

The first upper and lower jumps occur before a_1 and b_1 respectively, we have upper jump that occurs between a_k and a_{k+1} and lower jump that occurs between b_l and b_{l+1}.

Case 1: if $a_k = b_l$ and $a_{k+1} = b_{l+1}$, there is a jump before and after a_k, α can be expressed as

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_k & a_{k+1} & . & . & . & a_r \\ 1 & 2 & . & . & . & k & k+2 & . & . & . & n-1 \end{pmatrix}\begin{pmatrix} 1 & 2 & . & . & . & k & k+2 & . & . & . & n-1 \\ b_1 & b_2 & . & . & . & b_l & b_{l+1} & . & . & . & b_r \end{pmatrix}$$

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_k & a_{k+1} & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_l & b_{l+1} & . & . & . & b_r \end{pmatrix} \tag{33}$$

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix} \tag{34}$$

Case 2: if $a_k < b_l$, $a_{k+1} = b_{l+1}$, with a lower jump of length one that occurs between b_l and b_{l+1} and of length two before a_{k+1}.

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_k & a_{k+1} & . & . & . & a_r \\ 1 & 2 & . & . & . & k & k+3 & . & . & . & n-1 \end{pmatrix}\begin{pmatrix} 1 & 2 & . & . & . & k & k+3 & . & . & . & n-1 \\ b_1 & b_2 & . & . & . & b_l & b_{l+1} & . & . & . & b_r \end{pmatrix}$$

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_k & a_{k+1} & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_l & b_{l+1} & . & . & . & b_r \end{pmatrix} \tag{35}$$

In this case (case 2: $a_k < b_l$, $a_{k+1} = b_{l+1}$) , the first part is not fixed, but the second part is fixed.

If $a_{k+1} < b_{l+1}$, with having an upper jump of length one after a_{k+1}.

So,

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_k & a_{k+1} & a_{k+3} & . & . & . & a_r \\ 1 & 2 & . & . & . & k & k+3 & k+5 & . & . & . & n-1 \end{pmatrix}\begin{pmatrix} 1 & 2 & . & . & . & k & k+3 & k+5 & . & . & . & n-1 \\ b_1 & b_2 & . & . & . & b_l & b_{l+1} & b_{l+2} & . & . & . & b_r \end{pmatrix}$$

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_k & a_{k+1} & a_{k+3} & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_l & b_{l+1} & b_{l+2} & . & . & . & b_r \end{pmatrix} \tag{36}$$

Observe that from the transformation , $a_{k+3} = b_{l+2}$.

Case 3: if $a_k > b_l$;

Proof: Then α^{-1} is case 2 That is, the proof is similar to that of case 2.
So,

$$\alpha^{-1} = \begin{pmatrix} b_1 & b_2 & . & . & . & b_r \\ a_1 & a_2 & . & . & . & a_r \end{pmatrix}, \quad (37)$$

$$\alpha^{-1} = n_1 n_2, \quad (38)$$

Then

$$(\alpha^{-1})^{-1} = \alpha = n_2^{-1} n_1^{-1}, \text{ (since, } IO_n \text{ is an inverse semigroup)} \quad (39)$$

where $n_1, n_2 \in N$, N is a set of nilpotents of IO_n, IO_n is a partial one-to-one order-preserving transformation semigroup.

and if $a_{k+1} > b_{l+1}$, then

$$\alpha^{-1} = \begin{pmatrix} b_1 & b_2 & . & . & . & b_r \\ a_1 & a_2 & . & . & . & a_r \end{pmatrix} \quad (40)$$

$$\alpha^{-1} = n_1 n_2 \quad (41)$$

and then,

$$(\alpha^{-1})^{-1} = \alpha = n_2^{-1} n_1^{-1} \text{ (since, } IO_n \text{ is an inverse semigroup)} \quad (42)$$

(2) $a_1 \neq 1, b_1 \neq 1, a_r \neq n, b_r = n$, with at least lower jump of length greater than one and there exist an upper jump and a lower jump of length one before a_1 and b_1, then α can be expressed as

$$\alpha = \begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ 1 & 2 & . & . & . & r \end{pmatrix} \begin{pmatrix} 1 & 2 & . & . & . & r \\ c_1 & c_2 & . & . & . & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & . & . & . & c_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix} \quad (43)$$

$$c_i = \begin{cases} b_i + 1, & 1 \leq i \leq k \\ b_i - 1, & i > k \end{cases} \quad (44)$$

Assume that, the lower jump of length greater than one occurs between b_k and b_{k+1}.
So,

$$c_{k+1} = b_{k+1} - 1 \geq b_k + 3 - 1 = b_k + 2 > b_k + 1 = c_k \quad (45)$$

$$\Rightarrow c_{k+1} > c_k \quad (46)$$

$$\Rightarrow c_k < c_{k+1}. \quad (47)$$

$$\text{So, for all } i,\ c_i < c_{i+1} \quad (48)$$

(3) $a_1 \neq 1, b_1 \neq 1, a_r = n, b_r \neq n$, with at least one upper jump of length greater than one
From this we can see that α^{-1} is (2), so we will be applying similar approach to achieve (3). Then,

$$\alpha^{-1} = \begin{pmatrix} b_1 & b_2 & . & . & . & b_r \\ a_1 & a_2 & . & . & . & a_r \end{pmatrix}, \quad (49)$$

That is,

$$\alpha^{-1} = n_1 n_2 n_3 \tag{50}$$

and so,

$$(\alpha^{-1})^{-1} = \alpha = n_3^{-1} n_2^{-1} n_1^{-1}, \text{ (since, } IO_n \text{ is an inverse semigroup)} \tag{51}$$

where $n_1, n_2, n_3 \in N$, N is the set of nilpotents of IO_n, IO_n is the partial one-to-one order-preserving transformation semigroup.

(4) $a_1 \neq 1, b_1 \neq 1, a_r \neq n, b_r \neq n$, with an upper jump and a lower jump before a_1 and b_1.

So, α can be expressed as

$$\alpha = \begin{pmatrix} a_1 & . & . & . & a_r \\ 1 & . & . & . & r \end{pmatrix} \begin{pmatrix} 1 & . & . & . & r \\ b_1 & . & . & . & b_r \end{pmatrix} \tag{52}$$

Lemma 3.4: A transformation $\alpha \in IS_n$, where IS_n is partial one-one transformation semigroup, which has a nilpotent path and a permutation part is not nilpotent.

Proof: Suppose that $\alpha \in IS_n$ has a nilpotent path and a permutation part. Then there exists $x \in dom(\alpha)$ such that $x\alpha \notin dom(\alpha)$ and there exists some $x \in dom(\alpha)$ which form a cycle. The transformation can have the form:

$$\alpha = \begin{pmatrix} x_1 & x_2 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_5 & x_6 & x_4 \end{pmatrix} = (x_1 x_2 x_3](x_4 x_5 x_6) \tag{53}$$

That is,

$$\begin{pmatrix} x_1 & x_2 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_5 & x_6 & x_4 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_5 & x_6 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_4 & x_5 & x_6 \\ x_3 & \emptyset & x_6 & x_4 & x_5 \end{pmatrix} \tag{54}$$

$$\begin{pmatrix} x_1 & x_2 & x_4 & x_5 & x_6 \\ x_3 & \emptyset & x_6 & x_4 & x_5 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_5 & x_6 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_4 & x_5 & x_6 \\ \emptyset & \emptyset & x_4 & x_5 & x_6 \end{pmatrix} \tag{55}$$

$$\begin{pmatrix} x_4 & x_5 & x_6 \\ x_4 & x_5 & x_6 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_5 & x_6 & x_4 \end{pmatrix} = \begin{pmatrix} x_4 & x_5 & x_6 \\ x_5 & x_6 & x_4 \end{pmatrix} \tag{56}$$

$$\begin{pmatrix} x_4 & x_5 & x_6 \\ x_5 & x_6 & x_4 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_5 & x_6 & x_4 \end{pmatrix} = \begin{pmatrix} x_4 & x_5 & x_6 \\ x_6 & x_4 & x_5 \end{pmatrix} \tag{57}$$

$$\begin{pmatrix} x_4 & x_5 & x_6 \\ x_6 & x_4 & x_5 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_4 & x_5 & x_6 \\ x_2 & x_3 & x_5 & x_6 & x_4 \end{pmatrix} = \begin{pmatrix} x_4 & x_5 & x_6 \\ x_4 & x_5 & x_6 \end{pmatrix} \tag{58}$$

We have shown that for some power of α, that is, α^k, where k is some positive integer, for all $x \in dom(\alpha^k), x\alpha^k = x$ and hence α is not nilpotent(since $\alpha^k \neq \emptyset$, where k is some positive integer).

Corollary 3.4: A partial order-preserving transformation α, is defined by

$$\alpha = \begin{pmatrix} A_1 & A_2 & . & . & . & A_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix}$$

where for each $a_i \in A_i, a_i < a_{i+1}$ $(i = 1, 2, \ldots, r)$ and $b_1 < b_2 < \ldots < b_r$. Let $x_i = min\{x: x \in A_i\}$ and $y_i = max\{x: x \in A_i\}$. For $i = 1, 2, \ldots, r$, let $S_i = \{x \in X_n: x_i \leq x \leq y_i\}$, and for $r = 1, 2, \ldots, r-1$, $T_i = \{x \in X_n: y_i < x < x_{i+1}\}$. Let $T_0 = \{x \in X_n: x < x_1\}$ and $T_r = \{x \in X_n: x > y_r\}$.

The transformation α can be expressed as a product of nilpotents, if it satisfies one of the following:

(1) $1 \notin A_1, b_1 \neq 1, n \in A_r, b_r \neq n,$

(2) $1 \notin A_1, b_1 \neq 1, n \notin A_r, b_r = n,$

(3) $1 \notin A_1, b_1 \neq 1, n \notin A_r, b_r \neq n,$

(4) $1 \notin A_1, b_1 \neq 1, n \in A_r, b_r = n.$

Proof:

(1) Case 1: $1 \notin A_1, b_1 \neq 1, n \in A_r, b_r \neq n$, with a lower jump of length greater than or equal to two before x_i ,

α is given by

$$\begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix} \tag{59}$$

(a) Then $c \in S_i \setminus A_i$,

Therefore,

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_{i-1} & A_i & A_{i+1} & \ldots & A_r \\ x_1 & x_2 & \ldots & x_i & c & y_i & \ldots & y_{r-1} \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \ldots & x_i & c & y_i & \ldots & y_{r-1} \\ b_1 & b_2 & \ldots & b_{i-1} & b_i & b_{i+1} & \ldots & b_r \end{pmatrix}$$

(b) Then for $|T_i| \geq 2$, that is, there exists $c, d \in T_i$ such that $c < d$, with a lower jump of length greater than or equal to two before x_i.

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \ldots & A_r \\ x_1 & x_2 & \ldots & x_i & c & d & y_{i+1} & \ldots & y_{r-1} \end{pmatrix} \begin{pmatrix} x_1 & \ldots & x_i & c & d & y_{i+1} & \ldots & y_{r-1} \\ b_1 & \ldots & b_{i-1} & b_i & b_{i+1} & b_{i+2} & \ldots & b_r \end{pmatrix}$$

(2) $1 \notin A_1, b_1 \neq 1, n \notin A_r, b_r = n$, with at least one lower jump of length greater than one, which occurs between b_k and b_{k+1}, the first set A_1 contains more than one element.

$$\alpha = \begin{pmatrix} A_1 & \ldots & A_r \\ 1 & \ldots & r \end{pmatrix} \begin{pmatrix} 1 & \ldots & r \\ c_1 & \ldots & c_r \end{pmatrix} \begin{pmatrix} c_1 & \ldots & c_r \\ b_1 & \ldots & b_r \end{pmatrix} \tag{60}$$

$$c_i = \begin{cases} b_i + 1, & 1 \leq i \leq k \\ b_i - 1, & i > k \end{cases} \tag{61}$$

$$c_{k+1} = b_{k+1} - 1 \geq b_k + 3 - 1 = b_k + 2 > b_k + 1 = c_k \tag{62}$$

$$\Rightarrow c_{k+1} > c_k \tag{63}$$

$$\Rightarrow c_i < c_{i+1} \tag{64}$$

(3) $1 \notin A_1, b_1 \neq 1, n \notin A_r, b_r \neq n.$

$$\alpha = \begin{pmatrix} A_1 & \ldots & A_r \\ 1 & \ldots & r \end{pmatrix} \begin{pmatrix} 1 & \ldots & r \\ b_1 & \ldots & b_r \end{pmatrix} \tag{65}$$

(4) $1 \notin A_1, b_1 \neq 1, n \in A_r, b_r = n$, with at least one lower jump of length greater than one.

(a) For $c \in S_i \setminus A_i$, (that is, $S_i \neq A_i$.)

$$\alpha = \begin{pmatrix} A_1 & . & . & . & A_{i-1} & A_i & A_{i+1} & . & . & . & A_r \\ x_1 & . & . & . & x_i & c & y_i & . & . & . & y_{r-1} \end{pmatrix} \begin{pmatrix} x_1 & . & . & . & x_i & c & y_i & . & . & . & y_{r-1} \\ 1 & . & . & . & i-1 & i & i+1 & . & . & . & r \end{pmatrix}$$

$$\begin{pmatrix} 1 & . & . & . & i-1 & i & i+1 & . & . & . & r \\ c_1 & . & . & . & c_{i-1} & c_i & c_{i+1} & . & . & . & c_r \end{pmatrix} \begin{pmatrix} c_1 & . & . & . & c_{i-1} & c_i & c_{i+1} & . & . & . & c_r \\ b_1 & . & . & . & b_{i-1} & b_i & b_{i+1} & . & . & . & b_r \end{pmatrix}$$

$$c_i = \begin{cases} b_i + 1, & 1 \leq i \leq k \\ b_i - 1, & i > k \end{cases} \tag{66}$$

$$c_{k+1} = b_{k+1} - 1 \geq b_k + 3 - 1 = b_k + 2 > b_k + 1 = c_k \tag{67}$$

$$\Rightarrow c_{k+1} > c_k \tag{68}$$

$$\Rightarrow c_i < c_{i+1} \tag{69}$$

(b) For $|T_i| \geq 2$, that is, $c, d \in T_i$ such that $c < d$, then α can be expressed as a product of nilpotents.

$$\alpha = \begin{pmatrix} A_1 & A_2 & . & . & . & A_{i-1} & A_i & A_{i+1} & A_{i+2} & . & . & . & A_r \\ y_2 & y_3 & . & . & . & y_i & c & d & y_{i+1} & . & . & . & y_{r-1} \end{pmatrix} \begin{pmatrix} y_2 & . & . & . & y_i & c & d & y_{i+1} & . & . & . & y_{r-1} \\ 1 & . & . & . & i-1 & i & i+1 & i+2 & . & . & . & r \end{pmatrix}$$

$$\begin{pmatrix} 1 & . & . & . & i-1 & i & i+1 & . & . & . & r \\ c_1 & . & . & . & c_{i-1} & c_i & c_{i+1} & . & . & . & c_r \end{pmatrix} \begin{pmatrix} c_1 & . & . & . & c_{i-1} & c_i & c_{i+1} & . & . & . & c_r \\ b_1 & . & . & . & b_{i-1} & b_i & b_{i+1} & . & . & . & b_r \end{pmatrix} \tag{70}$$

$$c_i = \begin{cases} b_i + 1, & 1 \leq i \leq k \\ b_i - 1\ , & i > k \end{cases} \tag{71}$$

$$c_{k+1} = b_{k+1} - 1 \geq b_k + 3 - 1 = b_k + 2 > b_k + 1 = c_k \tag{72}$$

$$\Rightarrow c_{k+1} > c_k \tag{73}$$

$$\Rightarrow c_i < c_{i+1}\ \text{, for all } i. \tag{74}$$

4. Conclusion

This work is about nilpotents in some subsemigroups of the finite partial transformation semigroup; specifically, the subsemigroups are semigroups of partial one-to-one order-preserving mappings and partial order-preserving transformations. Since, nilpotency is a structural property of fundamental importance, there is need for us to instigate results in this area of study. We have enriched this area of study by describing some elements of PO_n and IO_n in term of nilpotents. The subsemigroups PO_n and IO_n are regular and inverse semigroups, respectively. We do have semigroup structures in some disciplines such as Biology, Sociology and so on, which are regular, inverse, E-inversive in nature. Finally, we concluded that, it is possible to enrich the two sub-semigroups: namely, partial one-to-one order-preserving transformation semigroup and partial order-preserving transformation semigroup by describing some elements of PO_n and IO_n in term of nilpotents. The results established, to some extent addressing the problem of determining depth of the subsemigroup generated by nilpotent transformations of IO_n and PO_n. And also, help in addressing the problem of determining the nilpotent rank of a subsemigroup of IO_n. Descriptions uncovered some detailed aspects of the subsemigroup generated by nilpotents of IO_n and PO_n. The only subgroup of IO_n and PO_n is the trivial one. The beneficiaries are Beginners, Algebraists and Researchers.

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