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Some Fixed Point Theorems in Convex S-Metric Space

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ABSTRACT

In this paper, we introduce convex S-metric space. We also establish and prove some fixed point theorems for self-mappings satisfying certain contraction principles on a complete convex S-metric space. Some examples are presented to validate the originality and applicability of our results. Finally, we improve, generalize and extend some recent results.

1. Introduction

Every physical problem can be modeled mathematically. Such problem will probably, be an ODE or any other equations. Before such model can be used, it is advisable to verify if the solution(s) of the equation exists. In 1922, Banach came out with an alternative method to find the existence of solutions of ODE using metric space. It was an impressive discovery. Since then, metric space had been a center of attraction to many authors working in this area of study. Some of the authors tried to generalize or extend the usual metric space (see Gahler, 1963; Dhage, 1992; Branciari, 2000; Mustafa and Sims, 2006; Adewale *et. al.,* 2024 and other references). Takahashi, (1970) initiated the notion of convexity in metric spaces in 1970. He also studied some fixed point theorems for non-expansive mappings in the space. A convex metric space is a generalized metric space. Every normed space and cone Banach space are respectively convex metric space and convex complete metric space. Subsequently (Beg, 2001, Beg and Abbas, 2006, 2007; Chang *et al.*, 2004; Ciric,1993; Shimizu and Takahashi,1992; Tian, 2005; Ding, 1998; Adewale *et. al*. 2020). Many others studied fixed point theorems in both convex metric spaces and quasi convex metric spaces. In this study, we introduced convex S-metric space. We establish and prove some fixed point theorems for self-mappings satisfying certain contraction principles on a complete convex S-metric space. Some examples are presented to validate the originality and applicability of our results.

2. Preliminaries

Definition 2.1 (Adewale *et. al.,* 2024) Let X be a nonempty set and S: X × X × X → R+, a function satisfying the following properties: for all $x, y, z, a \in X$,

i. $S(x,y,z) = 0 \; iff \; x = y = z$.
ii. $S(x,y,z) \le S(x,x,a) + S(y,y,a) + S(z,z,a)$.

Then (X,S) is S-metric space.

Definition 2.2 (Moosae, 2012) Let $(X, \|\|)$ be a normed space. The mapping $W : X \times X \times [0,1) \to X$ defined by $W(x,y,\lambda) = \lambda x + (1-\lambda)y$ for each $x, y \in X, \lambda \in [0,1)$ is a convex structure on X.

Definition 2.3 (Moosae, 2012) let (X, d, W) be a convex metric space. A nonempty subset C of X is said to be

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convex if $W(x, y, \lambda) \in C$ whenever $(x, y, \lambda) \in C \times C \times [0,1)$.

3. Main results

Definition 3.1 Let (X, S) be $S-$metric space and $\alpha + \beta + \gamma = 1$. A mapping $\theta : X \times X \times X \times [0,1] \times [0,1] \times [0,1] \to X$ is said to be a convex structure on X if for each $(x, y, z, \alpha, \beta, \gamma) \in X \times X \times X \times [0,1] \times [0,1] \times [0,1]$ and $u, v \in X$,

$$S(u, v, \theta(x, y, z, \alpha, \beta, \gamma)) \leq \alpha S(u, v, x) + \beta S(u, v, y) + \gamma S(u, v, z). \quad (1)$$

Lemma 3.2 Let (X, S, θ) be a convex S-metric space, then the following statements hold:

i. $S(x, y, x) \leq S(y, y, x), S(y, x, x) \leq S(y, y, x)$ *and* $S(x, x, y) \leq S(y, y, x)$
ii. $S(x, x, \theta(x, y, y, \alpha, \beta, \gamma)) + S(y, y, \theta(x, y, y, \alpha, \beta, \gamma)) \leq S(x, x, y)$.
iii. $S(x, x, \theta(x, y, y, \alpha, \beta, \gamma)) = (\beta + \gamma)S(x, x, y)$
iv. $S(y, y, \theta(x, y, y, \alpha, \beta, \gamma)) = \alpha S(y, y, x)$
v. $S(x, y, \theta(x, y, y, \alpha, \beta, \gamma)) \leq (\alpha + 2\beta + 2\gamma)S(y, y, x)$

Proofs:

i. $S(x, y, x) \leq S(x, x, x) + S(y, y, x) + S(x, x, x) = S(y, y, x)$.

$S(y, x, x) \leq S(y, y, x) + S(x, x, x) + S(x, x, x) = S(y, y, x)$.
$S(x, x, y) \leq S(x, x, x) + S(x, x, x) + S(y, y, x) = S(y, y, x)$.

ii. Let $S_x = S(x, x, \theta(x, y, y, \alpha, \beta, \gamma))$ and $S_y = S(y, y, \theta(x, y, y, \alpha, \beta, \gamma))$, (2)

then

$$Sx + Sy \leq \alpha S(x, x, x) + \beta S(x, x, y) + \gamma S(x, x, y)(2) + \alpha S(y, y, x) + \beta S(y, y, y) + \gamma S(y, y, y) \quad (3)$$
$$= \beta S(x, x, y) + \gamma S(x, x, y) + \alpha S(y, y, x) \quad (4)$$
$$\leq \beta S(x, x, y) + \gamma S(x, x, y) + \alpha S(x, x, y) \quad (5)$$
$$= (\alpha + \beta + \gamma)S(x, x, y). \quad (6)$$
$$= S(x, x, y) \quad (7)$$

iii. $S(x, x, \theta(x, y, y, \alpha, \beta, \gamma)) = \alpha S(x, x, x) + \beta S(x, x, y) + \gamma S(x, x, y)(8)$

$$= \beta S(x, x, y) + \gamma S(x, x, y) = (\beta + \gamma)S(x, x, y). \quad (8)$$

iv. $S(y, y, \theta(x, y, y, \alpha, \beta, \gamma)) = \alpha S(y, y, x) + \beta S(y, y, y) + \gamma S(y, y, y)(11)$

$$= \alpha S(y, y, x). \quad (9)$$

v. $S(x, y, \theta(x, y, y, \alpha, \beta, \gamma)) \leq \alpha S(x, y, x) + \beta S(x, y, y) + \gamma S(x, y, y)$

$$\leq \alpha S(x, x, y) + \beta S(x, y, y) + \gamma S(x, y, y)$$
$$\leq (\alpha + 2\beta + 2\gamma)S(y, y, x). \quad (10)$$

Definition 3.3 Let (X, S, θ) be a convex S-metric space. For $y \in X, r > 0$, the convex S-sphere with centre y and radius r is $S_r(y) = \{z \in X : S(z, z, \theta(y, z, z, \alpha, \beta, \gamma)) < r\}$.

Definition 3.4 Let (X, S, θ) be a convex S-metric space. A sequence $\{x_n\} \subset X$ is convex S-convergent to z if the limit of $S(x_n, z, \theta(z, z, z, \alpha, \beta, \gamma))$ tends to zero as n tends to infinity.

Definition 3.5 Let (X, S, θ) be a convex S-metric space. A sequence $\{x_n\} \subset X$ is said to be a convex S-Cauchy sequence if the limit of $S(x_n, x_m, \theta(x_l, x_l, x_l, \alpha, \beta, \gamma))$ tends to zero as n, m, l tends to infinity.

Definition 3.6 Let (X, S, θ) be a convex S-metric space and E be a convex subset of X. A self-mapping T on E has a property (I_s) if $T(\theta(x, y, z, \alpha, \beta, \gamma)) = \theta(T(x), T(y), T(z), \alpha, \beta, \gamma)$ for each $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha = 1 - \beta - \gamma$.

Example 3.7 If $(X, \|\|)$ is a Banach space, then every affine mapping on a convex subset of X has the property (I_s).
Example 3.8 Let $X = R$ and the convex S-metric be defined by $S(x,y,z,\theta(x,y,z,\alpha,\beta,\gamma)) = \alpha|x-y| + \beta|y-z| + \gamma|z-x|$, then (X,S,θ) is a convex S-metric space.
Theorem 3.9 Let X be a complete convex S-metric space and $T : X \to X$ a map for which there exist the real number, a satisfying $0 \leq a < 1$ such that for all $x, y \in X$.

$$S(T\,x, T\,y, \theta(T\,z, T\,z, T\,z, \alpha,\beta,\gamma)) \leq aS(x,y,\theta(z,z,z,\alpha,\beta,\gamma)) \tag{11}$$

Then T has a unique fixed point.

Proof:

Suppose T satisfies condition (11) and $x_o \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_o$, then

$$\begin{aligned} S(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha,\beta,\gamma)) &= S(Tx_{n-1}, Tx_n, \theta(T\ x_n, T\ x_n, T\ x_n, \alpha,\beta,\gamma)) \\ &\leq aS(x_{n-1}, x_n, \theta(x_n, x_n, x_n, \alpha,\beta,\gamma)) \\ &= a\alpha S(x_{n-1}, x_n, x_n) + a\beta S(x_{n-1}, x_n, x_n) + a\gamma S(x_{n-1}, x_n, x_n) \\ &= a(\alpha+\beta+\gamma) S(x_{n-1}, x_n, x_n) \\ &= aS(x_{n-1}, x_n, x_n) \end{aligned} \tag{12}$$

Also,

$$\begin{aligned} S(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha,\beta,\gamma)) &= \alpha S(x_n, x_{n+1}, x_{n+1}) + \beta S(x_n, x_{n+1}, x_{n+1}) + \gamma S(x_n, x_{n+1}, x_{n+1}) \\ &= (\alpha+\beta+\gamma) S(x_n, x_{n+1}, x_{n+1}) \\ &= S(x_n, x_{n+1}, x_{n+1}) \end{aligned} \tag{13}$$

Then, (12) and (13) implies

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &\leq aS(x_{n-1}, x_n, x_n) \\ &\leq a^2 S(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\leq a^3 S(x_{n-3}, x_{n-2}, x_{n-2}) \leq \cdots \leq a^n S(x_0, x_1, x_1) \end{aligned} \tag{14}$$

Taking the limit of $S(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha,\beta,\gamma))$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha,\beta,\gamma)) = \lim_{n\to\infty} a^n S(x_0, x_1, x_1) = 0 \tag{15}$$

Using (ii) of Definition 3.1 repeatedly with $n < m < l$, we obtain:

$$\lim_{n\to\infty} S(x_n, x_m, \theta(x_l, x_l, x_l, \alpha,\beta,\gamma)) = 0 \tag{16}$$

So, $\{x_n\}$ is a convex S-Cauchy sequence.

By completeness of (X,S,θ), there exist $x_0 \in X$ such that x_n is convex S-convergent to x_0.

Suppose $Tx_0 \neq x_0$,

$$S(x_n, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha,\beta,\gamma)) \leq aS(x_{n-1}, x_0, \theta(x_0, x_0, x_0, \alpha,\beta,\gamma)). \tag{17}$$

Taking the limit as $n \to \infty$ and using the fact that function is convex S-continuous in its variables, we get

$$S(x_0, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha,\beta,\gamma)) \leq aS(x_0, x_0, \theta(x_0, x_0, x_0, \alpha,\beta,\gamma)). \tag{18}$$

Hence,

$$S(x_n, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha,\beta,\gamma)) \leq 0 \tag{19}$$

This is a contradiction. So, $Tx_0 = x_0$.

To show the uniqueness, suppose $x_1 \neq x_2$ is such that $Tx_1 = x_1$ and $Tx_2 = x_2$, then

$$S(Tx_1, Tx_2, \theta(Tx_2, Tx_2, Tx_2, \alpha,\beta,\gamma)) \leq aS(x_1, x_2, \theta(x_2, x_2, x_2, \alpha,\beta,\gamma)) \tag{20}$$

Since $Tx_1 = x_1$ and $Tx_2 = x_2$, we have

$$S(x_1, x_2, \theta(x_2, x_2, x_2, \alpha,\beta,\gamma)) \leq 0 \tag{21}$$

which implies that $x_1 = x_2$.

Remark 3.10 Convex S metric space is a generalization of S metric space. The convexity on S metric space allows more applications in the space.

Corollary 3.11 Let (X,S) be a convex S metric space, for all $x,y,z,a \in X$,

i. $S(x,y,\theta(z,z,z,\alpha,\beta,\gamma)) = 0$ iff $x = y = z$.
ii. $S(x,y,\theta(z,z,z,\alpha,\beta,\gamma)) \leq S(x,x,\theta(a,a,a,\alpha,\beta,\gamma)) + S(y,y,\theta(a,a,a,\alpha,\beta,\gamma)) + S(z,z,\theta(a,a,a,\alpha,\beta,\gamma))$.

Proof:

i. If $S(x,y,\theta(z,z,z,\alpha,\beta,\gamma)) = 0$, then
$\alpha S(x,y,z) + \beta S(x,y,z) + \gamma S(x,y,z) = 0$
$\Rightarrow (\alpha + \beta + \gamma)S(x,y,z) = 0$
$\Rightarrow S(x,y,z) = 0.$

Conversely,
If $x = y = z$, then,
$S(x,y,z) = 0$ (Since (X, S) is S metric space).
$S(x,y,z) = 0$
$\Rightarrow (\alpha + \beta + \gamma)S(x,y,z) = 0$
$\Rightarrow \alpha S(x,y,z) + \beta S(x,y,z) + \gamma S(x,y,z) = 0$
Hence, $S(x,y,\theta(z,z,z,\alpha,\beta,\gamma)) = 0.$

iii. $S(x,y,\theta(z,z,z,\alpha,\beta,\gamma)) = \alpha S(x,y,z) + \beta S(x,y,z) + \gamma S(x,y,z)$

$= (\alpha + \beta + \gamma)S(x,y,z)$
$= S(x,y,z).$ (22)

$S(x,x,\theta(a,a,a,\alpha,\beta,\gamma)) = \alpha S(x,x,a) + \beta S(x,x,a) + \gamma S(x,x,a)$
$= (\alpha + \beta + \gamma)S(x,x,a)$
$= S(x,x,a).$ (23)

$S(y,y,\theta(a,a,a,\alpha,\beta,\gamma)) = \alpha S(y,y,a) + \beta S(x,x,a) + \gamma S(y,y,a)$
$= (\alpha + \beta + \gamma)S(x,x,a)$
$= S(y,y,a).$ (24)

$S(z,z,\theta(a,a,a,\alpha,\beta,\gamma)) = \alpha S(z,z,a) + \beta S(z,z,a) + \gamma S(z,z,a)$
$= (\alpha + \beta + \gamma)S(z,z,a)$
$= S(z,z,a).$ (25)

(22), (23), (24) and (25) implies
$S(x,y,\theta(z,z,z,\alpha,\beta,\gamma)) \leq S(x,x,\theta(a,a,a,\alpha,\beta,\gamma)) + S(y,y,\theta(a,a,a,\alpha,\beta,\gamma)) + S(z,z,\theta(a,a,a,\alpha,\beta,\gamma)).$

Theorem 3.12 Let X be a complete convex S-metric space and $T : X \to X$ a map for which there exist the real number, k satisfying $0 \leq k < \frac{1}{2}$ such that for all $x, y \in X$,

$$S(T x,T y,\theta(T z,T z,T z,\alpha,\beta,\gamma)) \leq k[S(x,T x,\theta(T x,T x,T x,\alpha,\beta,\gamma)) + S(y,T y,\theta(T y,T y,T y,\alpha,\beta,\gamma))] \quad (26)$$

Then T has a unique fixed point.

Proof:

Suppose T satisfies condition (26) and $x_o \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_o$, then

$S(x_n,x_{n+1},\theta(x_{n+1},x_{n+1},x_{n+1},\alpha,\beta,\gamma)) = S(Tx_{n-1},Tx_n,\theta(T x_n,T x_n,T x_n,\alpha,\beta,\gamma))$
$\leq k[S(x_{n-1},x_n,\theta(x_n,x_n,x_n,\alpha,\beta,\gamma)) + S(x_n,x_{n+1},\theta(x_{n+1},x_{n+1},x_{n+1},\alpha,\beta,\gamma))]$
$\leq \dfrac{k}{1-k} S(x_{n-1},x_n,\theta(x_n,x_n,x_n,\alpha,\beta,\gamma))$

Setting $a = \frac{k}{1-k}$, we have

$$S(x_n,x_{n+1},x_{n+1}) \leq aS(x_{n-1},x_n,x_n) \quad (27)$$

Also,
$S(x_n,x_{n+1},\theta(x_{n+1},x_{n+1},x_{n+1},\alpha,\beta,\gamma)) = \alpha S(x_n,x_{n+1},x_{n+1}) + \beta S(x_n,x_{n+1},x_{n+1}) + \gamma S(x_n,x_{n+1},x_{n+1})$
$= (\alpha + \beta + \gamma)S(x_n,x_{n+1},x_{n+1})$
$= S(x_n,x_{n+1},x_{n+1})$ (28)

(27) and (28) implies

$S(x_n, x_{n+1}, x_{n+1}) \leq aS(x_{n-1}, x_n, x_n)$
$\leq a^2 S(x_{n-2}, x_{n-1}, x_{n-1})$
$\leq a^3 S(x_{n-3}, x_{n-2}, x_{n-2}) \leq \cdots \leq a^n S(x_0, x_1, x_1)$ (29)

Taking the limit of $S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big)$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big) = \lim_{n\to\infty} a^n S(x_0, x_1, x_1) = 0 \quad (30)$$

Using (ii) of Definition 3.1 repeatedly with $n < m < l$, we obtain:

$$\lim_{n\to\infty} S\big(x_n, x_m, \theta(x_l, x_l, x_l, \alpha, \beta, \gamma)\big) = 0 \quad (31)$$

So, $\{x_n\}$ is a convex S-Cauchy sequence.

By completeness of (X, S, θ), there exist $x_0 \in X$ such that x_n is convex S-convergent to x_0.

Suppose $Tx_0 \neq x_0$,

$S(x_n, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)) \leq k[S\big(x_{n-1}, x_0, \theta(x_0, x_0, x_0, \alpha, \beta, \gamma)\big) +$
$S\big(x_0, x_0, \theta(x_0, x_0, x_0, \alpha, \beta, \gamma)\big)].$ (32)

Taking the limit as $n \to \infty$ and using the fact that function is convex S-continuous in its variables, we get

$S(x_0, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)) \leq 2kS(x_0, x_0, \theta(x_0, x_0, x_0, \alpha, \beta, \gamma)).$ (33)

Hence,

$S\big(x_n, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)\big) \leq 0$ (34)

This is a contradiction. So, $Tx_0 = x_0$.

To show the uniqueness, suppose $x_1 \neq x_2$ is such that $Tx_1 = x_1$ and $x_2 = x_2$,
then

$S\big(Tx_1, Tx_2, \theta(Tx_2, Tx_2, TTx_2, \alpha, \beta, \gamma)\big)$
$\leq k[S\big(x_1, Tx_1, \theta(Tx_1, Tx_1, Tx_1, \alpha, \beta, \gamma)\big)$
$+ S\big(x_2, Tx_2, \theta(Tx_2, Tx_2, TTx_2, \alpha, \beta, \gamma)\big)]$ (35)

Since $Tx_1 = x_1$ and $Tx_2 = x_2$, we have

$S\big(x_1, x_2, \theta(x_2, x_2, x_2, \alpha, \beta, \gamma)\big) \leq 0$ (36)

which implies that $x_1 = x_2$.

Theorem 3.13 Let X be a complete convex S-metric space and $T : X \to X$ a map for which there exist the real number, k satisfying $0 \leq k < \frac{1}{3}$ such that for all $x, y \in X$,

$S\big(T x, T y, \theta(T z, T z, T z, \alpha, \beta, \gamma)\big) \leq k\big[S\big(x, T x, \theta(T x, T x, T x, \alpha, \beta, \gamma)\big) +$
$S\big(y, T y, \theta(T y, T y, T y, \alpha, \beta, \gamma)\big) + S\big(z, T z, \theta(T z, T z, T z, \alpha, \beta, \gamma)\big)\big]$ (37)

Then T has a unique fixed point.

Proof:

Suppose T satisfies condition (37) and $x_o \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_o$, then

$S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big) = S\big(Tx_{n-1}, Tx_n, \theta(T x_n, T x_n, T x_n, \alpha, \beta, \gamma)\big)$
$\leq k[S\big(x_{n-1}, x_n, \theta(x_n, x_n, x_n, \alpha, \beta, \gamma)\big) + 2S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big)]$
$\leq \dfrac{k}{1-k} S\big(x_{n-1}, x_n, \theta(x_n, x_n, x_n, \alpha, \beta, \gamma)\big)$

Setting $a = \frac{k}{1-2k}$, we have

$S(x_n, x_{n+1}, x_{n+1}) \leq aS(x_{n-1}, x_n, x_n)$ (38)

Also,

$S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big) = \alpha S(x_n, x_{n+1}, x_{n+1}) + \beta S(x_n, x_{n+1}, x_{n+1}) + \gamma S(x_n, x_{n+1}, x_{n+1})$
$= (\alpha + \beta + \gamma) S(x_n, x_{n+1}, x_{n+1})$
$= S(x_n, x_{n+1}, x_{n+1})$ (39)

(38) and (39) implies

$S(x_n, x_{n+1}, x_{n+1}) \leq aS(x_{n-1}, x_n, x_n)$
$\leq a^2 S(x_{n-2}, x_{n-1}, x_{n-1})$
$\leq a^3 S(x_{n-3}, x_{n-2}, x_{n-2}) \leq \cdots \leq a^n S(x_0, x_1, x_1)$ (40)

Taking the limit of $S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big)$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big) = \lim_{n\to\infty} a^n S(x_0, x_1, x_1) = 0 \quad (41)$$

Using (ii) of Definition 3.1 repeatedly with $n < m < l$, we obtain:

$$\lim_{n\to\infty} S\big(x_n, x_m, \theta(x_l, x_l, x_l, \alpha, \beta, \gamma)\big) = 0 \quad (42)$$

So, $\{x_n\}$ is a convex S-Cauchy sequence.

By completeness of (X, S, θ), there exist $x_0 \in X$ such that x_n is convex S-convergent to x_0.

Suppose $Tx_0 \neq x_0$,

$$S(x_n, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)) \leq k[S\big(x_{n-1}, x_0, \theta(x_0, x_0, x_0, \alpha, \beta, \gamma)\big) + 2S\big(x_0, x_0, \theta(x_0, x_0, x_0, \alpha, \beta, \gamma)\big)]. \quad (43)$$

Taking the limit as $n \to \infty$ and using the fact that function is convex S-continuous in its variables, we get

$$S(x_0, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)) \leq 3kS(x_0, x_0, \theta(x_0, x_0, x_0, \alpha, \beta, \gamma)). \quad (44)$$

Hence,

$$S\big(x_n, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)\big) \leq 0 \quad (45)$$

This is a contradiction. So, $Tx_0 = x_0$.

To show the uniqueness, suppose $x_1 \neq x_2$ is such that $Tx_1 = x_1$ and $x_2 = x_2$, then

$$S\big(Tx_1, T(x_2), \theta(T(x_2), T(x_2), T(x_2), \alpha, \beta, \gamma)\big) \leq k[S\big(x_1, Tx_1, \theta(Tx_1, Tx_1, Tx_1, \alpha, \beta, \gamma)\big) + 2S\big(x_2, Tx_2, \theta(T(x_2), T(x_2), T(x_2), \alpha, \beta, \gamma)\big)] \quad (46)$$

Since $Tx_1 = x_1$ and $Tx_2 = x_2$, we have

$$S\big(x_1, x_2, \theta(x_2, x_2, x_2, \alpha, \beta, \gamma)\big) \leq 0 \quad (47)$$

which implies that $x_1 = x_2$.

Theorem 3.14 Let X be a complete convex S-metric space and $T : X \to X$ a map for which there exist the real number, k satisfying $0 \leq k < \frac{1}{3}$ such that for all $x, y \in X$, $S\big(T x, Ty, \theta(T z, T z, T z, \alpha, \beta, \gamma)\big) \leq k\big[S\big(x, T y, \theta(T y, T y, T y, \alpha, \beta, \gamma)\big) + S\big(y, T x, \theta(T x, T x, T x, \alpha, \beta, \gamma)\big)\big]$ (48)

Then T has a unique fixed point.

Proof:

Suppose T satisfies condition (48) and $x_o \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_o$, then

$$S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big) = S\big(Tx_{n-1}, Tx_n, \theta(Tx_n, Tx_n, Tx_n, \alpha, \beta, \gamma)\big)$$
$$\leq k[S\big(x_{n-1}, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big) + S\big(x_n, x_n, \theta(x_n, x_n, x_n, \alpha, \beta, \gamma)\big)]$$
$$\leq \frac{k}{1-2k} S\big(x_{n-1}, x_n, \theta(x_n, x_n, x_n, \alpha, \beta, \gamma)\big)$$

Setting $a = \frac{k}{1-2k}$, we have

$$S(x_n, x_{n+1}, x_{n+1}) \leq aS(x_{n-1}, x_n, x_n) \quad (49)$$

Also,

$$S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big) = \alpha S(x_n, x_{n+1}, x_{n+1}) + \beta S(x_n, x_{n+1}, x_{n+1}) + \gamma S(x_n, x_{n+1}, x_{n+1})$$
$$= (\alpha + \beta + \gamma) S(x_n, x_{n+1}, x_{n+1})$$
$$= S(x_n, x_{n+1}, x_{n+1}) \quad (50)$$

(49) and (50) implies

$$S(x_n, x_{n+1}, x_{n+1}) \leq aS(x_{n-1}, x_n, x_n)$$
$$\leq a^2 S(x_{n-2}, x_{n-1}, x_{n-1})$$
$$\leq a^3 S(x_{n-3}, x_{n-2}, x_{n-2}) \leq \cdots \leq a^n S(x_0, x_1, x_1) \quad (51)$$

Taking the limit of $S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big)$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big) = \lim_{n\to\infty} a^n S(x_0, x_1, x_1) = 0 \quad (52)$$

Using (ii) of Definition 3.1 repeatedly with $n < m < l$, we obtain:

$$\lim_{n\to\infty} S\big(x_n, x_m, \theta(x_l, x_l, x_l, \alpha, \beta, \gamma)\big) = 0 \quad (53)$$

So, $\{x_n\}$ is a convex S-Cauchy sequence.

By completeness of (X, S, θ), there exist $x_0 \in X$ such that x_n is convex S-convergent to x_0.

Suppose $Tx_0 \neq x_0$,

$$S(x_n, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)) \leq k[S\big(x_{n-1}, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)\big) + S\big(x_0, x_n, \theta(x_n, x_n, x_n, \alpha, \beta, \gamma)\big)]. \quad (54)$$

Taking the limit as $n \to \infty$ and using the fact that function is convex S-continuous in its variables, we get

$$S(x_0, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)) \leq kS(x_0, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)). \quad (55)$$

Hence,

$$S\big(x_0, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)\big) \leq 0 \quad (56)$$

This is a contradiction. So, $Tx_0 = x_0$.

To show the uniqueness, suppose $x_1 \neq x_2$ is such that $Tx_1 = x_1$ and $Tx_2 = x_2$, then

$$\begin{aligned} S\big(Tx_1, Tx_2, \theta(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma)\big) &\leq k[S\big(x_2, Tx_1, \theta(Tx_1, Tx_1, Tx_1, \alpha, \beta, \gamma)\big) \\ &+ S\big(x_1, Tx_2, \theta(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma)\big)] \end{aligned} \quad (57)$$

Since $Tx_1 = x_1$ and $T(x_2) = x_2$, we have

$$S\big(x_1, x_2, \theta(x_2, x_2, x_2, \alpha, \beta, \gamma)\big) \leq 0 \quad (58)$$

which implies that $x_1 = x_2$.

Theorem 3.15 Let X be a complete convex S-metric space and $T : X \to X$ a map for which there exist the real number, k satisfying $0 \leq k < \frac{1}{4}$ such that for all $x, y \in X$,

$$\begin{aligned} S\big(T x, T y, \theta(T z, T z, T z, \alpha, \beta, \gamma)\big) &\leq k\big[S\big(x, T y, \theta(T y, T y, T y, \alpha, \beta, \gamma)\big) + S\big(y, T z, \theta(T z, T z, T z, \alpha, \beta, \gamma)\big) \\ &+ S\big(z, T x, \theta(T x, T x, T x, \alpha, \beta, \gamma)\big)\big] \end{aligned} \quad (59)$$

Then T has a unique fixed point.

Proof:

Suppose T satisfies condition (59) and $x_o \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_o$, then

$$\begin{aligned} S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big) &= S\big(Tx_{n-1}, Tx_n, \theta(Tx_n, Tx_n, Tx_n, \alpha, \beta, \gamma)\big) \\ &\leq k[S\big(x_{n-1}, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big) + S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big)] \\ &\leq \frac{k}{1-3k} S\big(x_{n-1}, x_n, \theta(x_n, x_n, x_n, \alpha, \beta, \gamma)\big) \end{aligned}$$

Setting $a = \frac{k}{1-3k}$, we have

$$S(x_n, x_{n+1}, x_{n+1}) \leq aS(x_{n-1}, x_n, x_n) \quad (60)$$

Also,

$$\begin{aligned} S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big) &= \alpha S(x_n, x_{n+1}, x_{n+1}) + \beta S(x_n, x_{n+1}, x_{n+1}) + \gamma S(x_n, x_{n+1}, x_{n+1}) \\ &= (\alpha + \beta + \gamma) S(x_n, x_{n+1}, x_{n+1}) \\ &= S(x_n, x_{n+1}, x_{n+1}) \end{aligned} \quad (61)$$

(60) and (61) implies

$$\begin{aligned} S(x_n, x_{n+1}, x_{n+1}) &\leq aS(x_{n-1}, x_n, x_n) \\ &\leq a^2 S(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\leq a^3 S(x_{n-3}, x_{n-2}, x_{n-2}) \leq \cdots \leq a^n S(x_0, x_1, x_1) \end{aligned} \quad (62)$$

Taking the limit of $S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big)$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S\big(x_n, x_{n+1}, \theta(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)\big) = \lim_{n\to\infty} a^n S(x_0, x_1, x_1) = 0 \quad (63)$$

Using (ii) of Definition 3.1 repeatedly with $n < m < l$, we obtain:

$$\lim_{n\to\infty} S\big(x_n, x_m, \theta(x_l, x_l, x_l, \alpha, \beta, \gamma)\big) = 0 \quad (64)$$

So, $\{x_n\}$ is a convex S-Cauchy sequence.

By completeness of (X, S, θ), there exist $x_0 \in X$ such that x_n is convex S-convergent to x_0.

Suppose $Tx_0 \neq x_0$,

$$S(x_n, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)) \leq k[S(x_{n-1}, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)) + S(x_0, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)) + S(x_0, x_n, \theta(x_n, x_n, x_n, \alpha, \beta, \gamma))]. \quad (65)$$

Taking the limit as $n \to \infty$ and using the fact that function is convex S-continuous in its variables, we get

$$S(x_0, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)) \leq 2kS(x_0, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)). \quad (66)$$

Hence,

$$S(x_0, Tx_0, \theta(Tx_0, Tx_0, Tx_0, \alpha, \beta, \gamma)) \leq 0 \quad (67)$$

This is a contradiction. So, $Tx_0 = x_0$.

To show the uniqueness, suppose $x_1 \neq x_2$ is such that $Tx_1 = x_1$ and $x_2 = x_2$,

then

$$S(Tx_1Tx_2, \theta(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma)) \leq k[S(x_2, Tx_1, \theta(Tx_1, Tx_1, Tx_1, \alpha, \beta, \gamma)) + S(x_1, Tx_2, \theta(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma)) + S(x_2, Tx_2, \theta(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma))] \quad (68)$$

Since$Tx_1 = x_1$ and Tx_2, we have

$$S(x_1, x_2, \theta(x_2, x_2, x_2, \alpha, \beta, \gamma)) \leq 0 \quad (69)$$

which implies that $x_1 = x_2$.

4. Discussion

Convexity is a vital tool in a topological space. A non-empty set $E$is said to be convex if for any $x, y \in E, \gamma x + (1 - \gamma)y \in E, \gamma \in [0,1)$. In this paper, the concept of convexity was placed on S-metric space (a generalization of metric space). Convexity on a space, allows more applications if use to solve physical problems.

If convexity is removed from Theorem 3.9, it reduces to Banach Contraction Principle in S-metric space. If convexity is removed from Theorem 3.12, it reduces to Kannan Theorem in S-metric space. If convexity is removed from Theorem 3.13, 3.14 and 3.15, we obtain some theorems in S-metric space.

5. Conclusion

We introduce S-metric space with convexity in this paper. We also established some fixed point theorems for self-mappings satisfying certain contraction principles on a complete convex S-metric space. The proofs of these theorems are shown. Some examples were presented to validate the originality and applicability of our results. This work improves, generalizes and extends some recent results.

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