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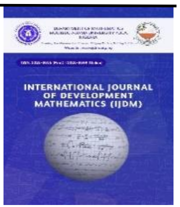

A Non-Standard Finite Difference Schemes for the Solution of Stiff Initial Value Problems

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ABSTRACT

In this study, we introduce a novel non-standard finite difference (NSFD) scheme designed to address the challenges posed by stiff initial value problems. Stiffness in differential equations often leads to numerical instability and requires specialized methods for stable and accurate solutions. A novel set of numerical schemes for solving stiff ordinary differential equations caused by the decay of radioactive substances developed. This paper demonstrates the power of normalization in the discretization function. We employed non-local approximation and renormalization of the denominator function to create qualitatively stable schemes for a stiff ordinary differential equation. The schemes' stability properties were verified using numerical experiments. The schemes' performance is evaluated in comparison to other typical finite difference schemes.

1. Introduction

Standard numerical techniques can pose significant challenges when approximating the solution of a differential equation, particularly when the exact solution involves complex terms of the form $e^{çt}$ with ç being a negative real number. However, for problems that involve rapidly decaying transient solutions, such as those common in spring and damping systems, control system analysis, and chemical kinetics, among others, Non-Standard Finite Difference Schemes (NSFDS) offer a compelling alternative. NSFDS have emerged as a reliable method for solving a broad range of mathematical models involving algebraic, differential, biological, and chaotic systems. These schemes have proven to be particularly effective for "stiff" systems of differential equations that arise in the analysis of spring and mass systems with large spring constants.

The non-standard finite difference method, introduced by Mickens (1994) over two decades ago, is an approach that leverages specific properties of the solution of the differential equations involved. Mickens' non-standard finite difference techniques have laid the foundation for designing methods that preserve the dynamics and stability of fixed points of the approximated differential system. Upon close examination of the differential equations for which exact schemes are known, it becomes clear that the denominator function is generally related to particular solutions or properties of the general solution to the differential equation (Ibijola *et al*., 2000).

Appadu (2013) employed three numerical methods to solve the one-dimensional advection-diffusion equation with constant coefficients, while Shabbir (2019) introduced a nonstandard finite difference scheme for a specific category of predator-prey systems with Holling type-III functional response. Also, Dang and Hoang (2019) converted a continuous-time predator-prey system with general functional response and recruitment for both species into a discrete-time model using a nonstandard finite difference (NSFD) scheme. Recent literature has focused on extending NSFD schemes to variable-order fractional problems, using techniques like the Grunwald-Letnikov and Riemann-Liouville derivatives to approximate variable order fractional-order equations (Sharma et. al, 2021). In our work, we will be employing the powerful non-local approximation technique proposed by Anguelov and Lubuma (2003). This

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non-local approximation will enable us to systematically select suitable free parameters in a way that the scheme satisfies the desired qualitative properties with precision and accuracy.

The initial value problem presented below undoubtedly transforms into a stiff equation when dealing with sufficiently small values $ç << 0$.

$$\boldsymbol{y''} = çy.\ y(0) = y_0, ç < 0 \qquad (1)$$

Have analytic property

$$y(t) = e^{çt},\ y(t) \to 0 \text{ as } t \to \infty \qquad (2)$$

2. Material and Methodology

The non-standard discrete model for an ODE of the form $y^g = f(x, y)$ can be written as $y_{n+1} = y_n + ɲf(x_n y_n)$

$$\Rightarrow \frac{y_{n+1} - y_n}{ɲ(\mathrm{h})} = f(x_n, y_n) \qquad (4)$$

The condition for the construction of the ɲ denominator requires that

$$ɲ(\mathrm{h}) = \mathrm{h} + 0(\ \mathrm{h}^2);\ \ \mathrm{h} \to 0 \qquad (5)$$

Approximate $çy$ in equation (1) by

$$ç(\mathrm{ay_k} + \mathrm{by_{k+1}}), \mathrm{a} + \mathrm{b} = 1, \mathrm{a}, \mathrm{b} \in \mathbb{R} \qquad (6)$$

From (3) we have

$$\frac{y_{k+1} - y_k}{ɲ(h)} = f(x_k y_k) \qquad (6)$$

$$y_{k+1} - y_k = çɲ(ay_k + by_{k+1}) \qquad (7)$$

$$y_{k+1} - çbɲy_{k+1} = y_k + açφɲ \qquad (8)$$

$$y_{k+1} = \frac{y_k(1 + açɲ)}{1 - çbɲ} \qquad [\mathrm{B1}] \qquad (9)$$

This scheme has been optimized with an ideal renormalized denominator function as

$$ɲ = \frac{(e^{\mathrm{xh}} - 1)}{\propto}, \propto \in \mathbb{R} \qquad (10)$$

$$ɲ(\mathrm{h}) = \mathrm{h} + 0(\ \mathrm{h}^2);\ \mathrm{h} \to 0$$

I. Using the Adams-Moulton two-stage method (9)

$$y_{k+1} = y_k + ç\frac{h}{z}(y_k + y_{k+1}) \qquad (11)$$

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$$y_{k+1} = \frac{y_k\left(1+\varsigma\frac{h}{2}\right)}{1-\varsigma\frac{h}{2}} \qquad [B2] \qquad (12)$$

II. Using Euler's forward difference method

$$y_{k+1} = y_k + \varsigma h_{y_k} \qquad (13)$$

2.1 Stability of the Schemes

The stability of the schemes can be proved by Fatunla (1998).

Let $y_n = y(x_n)$ and $p_n = p(x_n)$ denote two different numerical solution of the differential equation with the initial condition specified a $y_0 = y(x_0) = \xi$ and $p_0 = p(x_0) = \xi^*$ respectively such that $|\xi - \xi^*| < \varepsilon \quad \varepsilon > 0$

If the two numerical estimates are generated by the integration scheme, we have

$$y_{n+1} = y_n + h\phi(x_n, y_n, h)$$

$$p_{n+1} = p_n + h\phi(x_n, p_n, h)$$

The condition that $|y_{n+1} - p_{n+1}| \leq K \; |\xi - \xi^*|$ is the necessary and sufficient condition for the stability and convergence of the schemes.

For small 'h' the nonlocal approximation of

$$2y_n - y_{n-1} \cong y_n$$

The incremental function can be written as

$$\phi(x_n, y_n, h) = A + Bf_n'$$

The value of A is fixed for every finite ($n << \infty$)

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) = B[f'(x_n, y_n, h) - f'(x_n, y_n^*, h)]$$

$$= B[f'(x_n, y_n) - f'(x_n, y_n^*)]$$

$$= B[\frac{\partial f'(x_n, \bar{y})}{\partial y_n}(y_n - y_n^*)]$$

$$L = \mathrm{SUP}_{(x_n, y_n)\in D} \quad \frac{\partial f'(x_n, \bar{y})}{\partial y_n}$$

then,

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) = B[L(y_n - y_n^*)]$$

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) \leq M|y_n - y_n^*| \, .$$

which is the condition for convergence.

When $h = 0, A = 0 \quad B = 0$

$\Rightarrow y_{n+1} = y_n$ and the incremental function is identically zero when $h = 0$

$$\Rightarrow \phi(x_n, y_n, 0) \equiv 0$$

Consider the equation

$$y_{n+1} = y_n + \{A\} + \{B\}f_n'(x_n, y_n).$$

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Let $S_{n+1}= S_n+\{A\} +\{B\}f_n'(x_n, S_n)$,

$$y_{n+1} - S_{n+1}=y_n - S_n+\{A - A\}+\{B\}[f_n'(x_n, y_n) - f_n'(x_n, S_n)]$$

$$= y_n - S_n+ \ B[\frac{\partial f'(x_n,\ S_n)}{\partial p_n}(y_n - S_n)]$$

$$\text{L} = \text{SUP}_{(x_n,y_n)\in D} \ \frac{\partial f'(x_n,\ S_n)}{\partial p_n}.$$

$$y_{n+1} - S_{n+1}= y_n - S_n \ + B.L(y_n - \ S_n)$$

$$|y_{n+1} - S_{n+1}|= |y_n - S_n|+ [\,B.L\,]|(y_n - p_n)|.$$

$$\text{Let N} = |1+ [\,B.L\,]|$$

$$|y_{n+1} - S_{n+1}|\leq \text{N}\ |y_n - S_n|$$

Let $y_0= y(x_0) = \xi$ and $S= S(x_0) = \xi^*$ then,

$$|y_{n+1} - S_{n+1}|\leq \text{K} \ \ |\xi - \xi^*|.$$

3. Numerical Implementation

In this numerical experiment, we have carefully chosen the value of $\text{ç} = -15$ to achieve a specific level of stiffness and rapidly decaying transient solutions. We have developed highly efficient numerical algorithms to implement these schemes and precisely calculated the deviation error from the analytic solution. The obtained results are presented in Figures 1-4.

List of Schemes for $y' = \text{ç}y.\ y(0) = y_0$

	Scheme	Analytical Solution
1	$y_{k+1} = \frac{y_k(1 + a\text{çɲ})}{1 - b\text{çɲ}}$ $\text{ɲ} = \frac{(e^{\mathrm{xh}} - 1)}{\beta}, \mathrm{a} = 0.001, \mathrm{b} = 1 - \mathrm{a}, \beta = 15$	B1 [Eqn(9and10)]
2	$\mathrm{y_{k+1}} = \mathrm{y_k} + \text{ç}\frac{\mathrm{h}}{2}(\mathrm{y_k} + \mathrm{y_{k+1}})$	B2[(Eqn(12)]
3	$y_{k+1} = y_k + \lambda h y_k$	B3[(Eqn(13)]

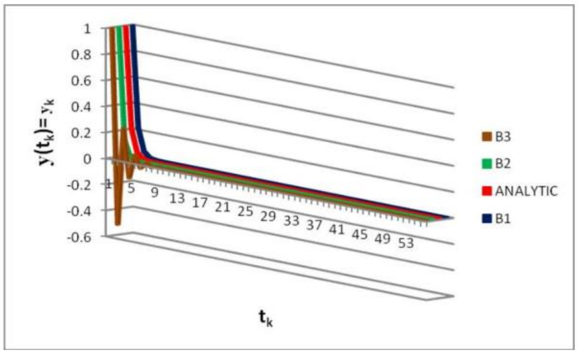

Figure 1: Graphical representation of the performance of the three schemes when $(h = 0.1)$

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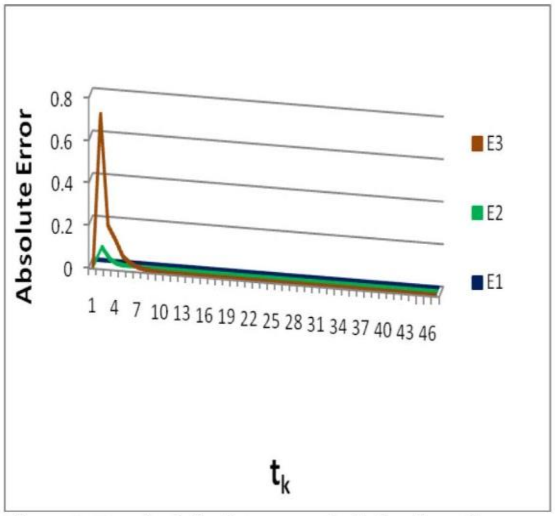

Figure 2: Graph of absolute error deviation from the analytic solution for $h = 0.01$

Values of Ei's in the graph above were obtained by taking the absolute deviation error of Scheme B1 from the analytic solution.

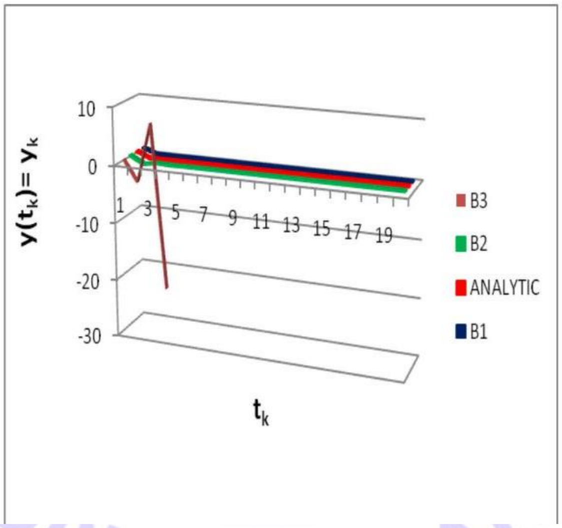

Figure 3: Graphical representation of performance of the schemes at off grid point $(h = 0.25)$

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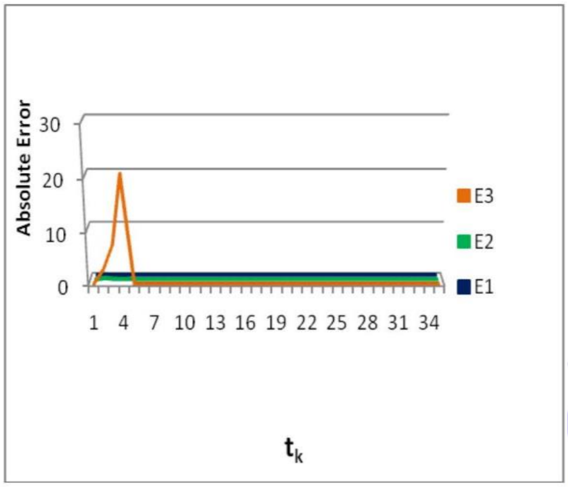

Figure 4: Graph of absolute error deviation from the analytic solution for $\mathbf{h = 0.25}$

4. Interpretation of the results

This research uncovered the potential of non-standard schemes when compared to two standard schemes. Through extensive numerical testing, we have found that the newly constructed schemes exhibit the same qualitative properties as the original ODE. The results of our study are presented in the graphs. Our findings emphasize the importance of proper renormalization of discretization functions, which can lead to the development of schemes that preserve the original ODE's qualitative behavior. With enough analytic information about an IVP, we can create schemes that are designed to maintain the ODE's unique characteristics. This research is a significant step forward in the development of numerical methods for solving IVPs, and we believe that it has the potential to greatly impact the field.

Euler's method fails when the step size is too large, as in Fig1 and 2. However, for lower step sizes, this scheme works well and the numerical results decrease to zero, just like the exact solution. The two-stage Adams-Moulton method with a step size of h=0.1 produces a solution that decreases to zero like the analytic solution, but oscillates about zero and the oscillation becomes wider in amplitude as the step size increases. . The choice of a suitable exponential function that satisfies the Micken rule has helped to reduce the rate of stiffness or variation. For scheme B1, the value of the parameter is carefully chosen as β=-ç, as long as ɲ follows the nonstandard modeling rules

5. Conclusion

The non-standard modeling rules developed by Mickens have proven to be a powerful tool for discrete modeling. They have led to the creation of various techniques for building numerically stable finite difference schemes. To solve the ODE, we have used the non-local approximation method introduced by Anguelov and Lubuma. This method enables us to systematically select appropriate free parameters to obtain a convex combination of more than one grid point as an approximation of the discretization function of the ODE. Additionally, it is supported by a suitably constructed denominator function that is specific to the ODE (10). This method has not only prevented the erratic behavior exhibited by standard schemes for this ODE but has also produced schemes that behave qualitatively and monotonically like the analytic solution.

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