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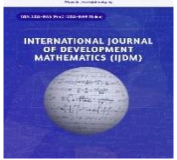

Numerical Solution of Generalized Delay Integro-Differential Equations via Galerkin-Vieta-Lucas Polynomials

Kazeem Issa[a][*], Muritala H. Sulaiman[a], Esther O. Olabamidele[a], and Ayinde M. Abdullahi[b]
[a]*Department of Mathematics and Statistics, Kwara State University, Ilorin, Nigeria.*
[b]*Department of Mathematics, University of Abuja, Abuja, Nigeria.*

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ABSTRACT

In this article, the Galerkin-Vieta-Lucas scheme is presented to find an approximate solution to the generalised delay integro-differential equation using the Vieta-Lucas polynomial as an approximation. The Galerkin approach transforms the delay integro-differential equation into a set of $n \times (n+m)$ algebraic equations, which, together with the attached conditions, give $(n+m) \times (n+m)$ equations. The effectiveness and accuracy of the proposed technique were tested on some existing examples in the literature, and obviously, the results obtained justify the accuracy of the proposed scheme.

1. Introduction

One of the equations gotten from models such as biological, physical, and engineering problems is the integro-differential equation (IDE), and much attention has been given to the approaches for solving the resulting equations. Some the approaches developed and adopted recently are, Homotopy approach (Dehghan and Shakeri, 2008); (Saberi-Nadjafi and Ghorbani, 2009), (Golbabai and Seifollahi, 2007); (Dzhumabaev, 2018) developed a generalized solutions for solving linear Fredholm IDEs, (Jangveladze *et al*., 2011); (Chen and Shih, 1997) employed finite element method, (Shahsavara, 2010) implemented Haar wavelets to test the accuracy of the solution, (Hosseini and Shahmorad, 2003a); (Hosseini and Shahmorad, 2003b); (Shahmorad, 2005) extended Tau method to IDE, (Issa and Salehi, 2017); (Biazar and Salehi, 2016); (Fathy *et al*., 2014) adopted Galerkin method to solve IDE, while perturbed-Galerkin method was introduced by (Issa *et al*., 2022), (Hou and Yang, 2013) used hybrid function operational matrix of derivative to approximate Fredholm IDE, (Oyedepo *et al*., 2024) employed the Vieta-Lucas polynomial to solve Volterra IDE.

[*] Corresponding author. Tel.: +2348036554437
E-mail address: issa.kazeem@kwasu.edu.ng (Kazeem I.)

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In this article, we consider generalized delay integro-differential equation (GDIDE) of the form

$$\begin{cases} \mathscr{D}y(x) + \int_0^x A_1(x,t)y(t-\delta)dt + \int_0^b A_2(x,t)y(t-\delta)dt = g(x), & x \in [0,\ b] \\ y(x) = \Phi(x), & x \in [-\delta,\ 0] \end{cases} \tag{1}$$

subject to the attached conditions:

$$\sum_{j=0}^{m-1}\sum_{r=1}^{i} \rho_{s,r}^{(j)} y^{(j)}(\tau_r) = \eta_s,\ s = 1, 2, \cdots,\ m \tag{2}$$

where δ is the delay term, $(x-\delta)$ is the delay argument, and $y(x-\delta)$ is the delay solution expression.

Several methods have been developed to solve Eq. (1) in the last decades: (Issa *et al*., 2019) adopted the shifted Chebyshev approach via the perturbed-Galerkin method; (Shahmorad and Ostadzad, 2016) developed the Tau method via the operational approach to solve Eq. (1); (Hetmaniok *et al*., 2022) proposed a procedure based on the Taylor differential transformation to transform Eq. (1) into a system of algebraic equations; stability and boundedness results to Volterra IDEs with delay were investigated by (Tunç, 2016); Eq. (1) was solved using the multistep block method involving two-point with constant step size by (Baharum *et al*., 2022). The aim of this article is to investigate the accuracy of the Vieta-Lucas polynomial as an approximation to solve GDIDEs and demonstrate the accuracy of the new techniques on some selected problems in the literature.

The other parts of this article are organised as follows: Section 2 defines some orthogonal polynomials; in Section 3, we present a numerical scheme. We implement the proposed scheme on some selected problems in Section 4, and finally, in Section 5, we have a discussion of the results and the concluding remarks.

2. Preliminaries

This section provides some important definitions and mathematical preliminaries of some orthogonal polynomials that are essential to this article.

2.1 υ-th kind shifted Chebyshev polynomials

Suppose $x = \left(\frac{2u-(a+b)}{b-a}\right)$, then the υ-th kind shifted Chebyshev polynomials are defined herein respect to their respective weight function $\omega(x)$, $x \in [a,b]$ as:

$$\Phi_i(x) = \begin{cases} \mathscr{T}_i(x) = \cos(ix), & \omega(x) = \dfrac{1}{\sqrt{1-x^2}},\ \upsilon = 1 \\ \mathscr{U}_i(x) = \dfrac{\sin\,(i+1)\,x}{\sin\,(x)}, & \omega(x) = \sqrt{1-x^2},\ \upsilon = 2 \\ \mathscr{V}_i(x) = \dfrac{\cos\left(i+\frac{1}{2}\right)x}{\cos\left(\frac{x}{2}\right)}, & \omega(x) = \sqrt{\dfrac{1+x}{1-x}},\ \upsilon = 3 \\ \mathscr{W}_i(x) = \dfrac{\sin\left(x+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}, & \omega(x) = \sqrt{\dfrac{1-x}{1+x}},\ \upsilon = 4 \end{cases} \tag{3}$$

See Issa *et al*. (2024) for details

2.2 Generalized Laguerre polynomials

Generalized Laguerre polynomial $\mathcal{L}_{\boldsymbol{i}}^{(\boldsymbol{\alpha})}(\boldsymbol{t}))$ reference to $\omega(t) = t^{\alpha}e^{-t}, \alpha > -1,\ t \in [0,\infty)$ is defined as follows (Daşcioġlu and Bayram, 2019):

$$\ell_i^{(\alpha)}(t) = \sum_{j=0}^{i} (-1)^j \binom{i+\alpha}{i-j} \frac{t^j}{j!} \tag{4}$$

The recurrence relation corresponding to Eq. (4) is defined as:

$$\ell_i^{(\alpha)}(t) = \frac{1}{i}\left[(\alpha + 2i - t - 1)\mathcal{L}_{i-1}^{(\alpha)}(t) - (\alpha + i - 1)\mathcal{L}_{i-2}^{(\alpha)}(t)\right], i \geq 2,$$

$$\mathcal{L}_0^{(\alpha)}(t) = 1,\ \mathcal{L}_1^{(\alpha)}(t) = \alpha + 1 - t \tag{5}$$

2.3 Ultraspherical Polynomials $\boldsymbol{\ell_m}^{(\boldsymbol{\alpha})}(\boldsymbol{t})$

Ultraspherical Polynomials $\boldsymbol{\ell_m}^{(\boldsymbol{\alpha})}(\boldsymbol{t})$ $t \in [-1, 1]$ with $\omega(t) = (1 - t^2)^{(\alpha - \frac{1}{2})}$ is defined as:

$$\ell_{\boldsymbol{m}}^{(\alpha)}(t) = \sum_{n=0}^{m} \frac{(-1)^n \Gamma(2\alpha + 2m - n)\Gamma\left(\alpha + \frac{1}{2}\right)}{(m-n)!\Gamma(2\alpha)\Gamma(n+1)\Gamma\left(m - n + \alpha + \frac{1}{2}\right)} t^{m-n} \tag{6}$$

With the recurrence relation:

$$\ell_m^{(\alpha)}(t) = \frac{1}{m}\left[2(m + \alpha - 1)t\ell_{m-1}^{(\alpha)}(t) - (m + 2\alpha - 2)\ell_{m-2}^{(\alpha)}(t)\right], m \geq 1, \ell_0^{(\alpha)}(t) = 1, \ell_1^{(\alpha)}(t) = 2\alpha t \tag{7}$$

The shifted form denoted by $\ell_m^{(\boldsymbol{\alpha})*}(\boldsymbol{t})$ $t \in [a, b]$ is defined as:

$$\ell_{m+1}^{(\alpha)*}(t) = \frac{1}{m+1}\left[2(m + \alpha)\left(\frac{2t-(a+b)}{b-a}\right)\ell_m^{(\alpha)*}(t) - (m + 2\alpha - 1)\ell_{m-1}^{(\alpha)*}(t)\right], m \geq 1 \tag{8}$$

where $\ell_m^{(\alpha)*}(x) = 1, \ell_m^{(\alpha)*}(x) = 2\alpha\left(\frac{2t-(a+b)}{b-a}\right)$.

The analytic form corresponding to Eq. (8) in the interval $u \in [0,\ 1]$ is given as:

$$\ell_{\boldsymbol{m}}^{(\alpha)*}(x) = \sum_{n=0}^{m} \frac{(-1)^n \Gamma(2\alpha + 2m - n)\Gamma\left(\alpha + \frac{1}{2}\right)}{(m-n)!\Gamma(2\alpha)\Gamma(n+1)\Gamma\left(m - n + \alpha + \frac{1}{2}\right)} x^{m-n} \tag{9}$$

The orthogonality relation corresponding to $\boldsymbol{\ell_m}^{(\boldsymbol{\alpha})*}(\boldsymbol{t})$ is given as:

$$\langle \mathscr{C}_m^{(\alpha)*}(t),\ \mathscr{C}_n^{(\alpha)*}(t)\rangle = \int_0^1 \left(t-t^2\right)^{\left(\alpha-\frac{1}{2}\right)} \mathscr{C}_m^{(\alpha)*}(t)\mathscr{C}_n^{(\alpha)*}(t)dt = \begin{cases} 0, & \text{for } m \neq n \\ \frac{\pi 2^{1-4\alpha}\Gamma(n+2\alpha)}{n![\Gamma(\alpha)]^2(n+\alpha)} & \text{, for } m = n \end{cases} \tag{10}$$

See (Issa *et al*., 2024) for more details.

2.4 Shifted Vieta-Lucas polynomials

Shifted Vieta-Lucas polynomials $V\ell_m^*(\boldsymbol{t})$ $t \in [a,b]$ with weight function $\omega(t) = \frac{b-a}{2\left(\sqrt{(b-a)^2-(2t-a-b)^2}\right)}$ is defined as

$$\mathscr{V}\ell_{i+1}^*(t) = \mathscr{V}\ell_1^*(t)\mathscr{V}\ell_i^*(t) - \mathscr{V}\ell_{i-1}^*(t), i \geq 1, \quad \mathscr{V}\ell_0^*(t) = 2, \mathscr{V}\ell_1^*(t) = \frac{4t-2(a+b)}{b-a}, \tag{11}$$

the analytic form corresponding to Eq. (11) in the interval $t \in [0,1]$ is given by:

$$\mathscr{V}\ell_i^*(t) = 2i\sum_{j=0}^{i} \frac{(-1)^j 4^{i-j}\Gamma(2i-j)}{\Gamma(j+1)\Gamma(2i-2j+1)} t^{i-j}. \tag{12}$$

The inner product of $V\ell_i^*(t$) and other properties are given in (Issa *et al*., 2024); (Agarwal and El-Sayed, 2020); (Youssef et. al., 2022).

3. Description of the Scheme

In this section, we propose the Galerkin-Vieta-Lucas scheme to find an approximate solution to Eq. (1). We begin replacing *y*(*x*) in Eq. (1) by approximating the polynomial

$$y_n(x) = \sum_{i=0}^{n} \alpha_i \mathscr{V}\ell_i^*(x), \quad x \in [a,b], \tag{13}$$

Such that Eq. (1) becomes:

$$\begin{cases} Dy_n(x) + \int_0^x A_1(x,t)y_n(t-\delta)dt + \int_0^b A_2(x,t)y_n(t-\delta)dt = g(x), & x \in [0,b] \\ y_n(x) = \Phi(x), & x \in [-\delta, 0], \end{cases} \tag{14}$$

where

$$D \equiv \sum_{r=0}^{m} P_r(x)\frac{d^r}{dx^r}$$

$P_r(x)$, *g*(*x*) and *A*(*x, t*) are known functions, *m* is the order of Eq. (1). We then multiply Eq. (14) by the shifted Vieta-Lucas $\mathscr{V}\ell_j^*(x),\ j = m, m+1, \cdots, n$ and integrate.

$$\int_{-\gamma}^{b} \left[\sum_{q=0}^{m} P_q(x) \sum_{i=0}^{n} \left(\frac{4}{b-a}\right)^q \alpha_i \frac{d^q}{dx^q}\left(\mathscr{V}\ell_i^*(x)\right) + \int_0^x A_1(x,t)y_n(t-\delta)dt + \int_0^b A_2(x,t)y_n(t-\delta)dt + + \int_0^\delta A_2(x,t)\Phi(t)\right] \mathscr{V}\ell_j^*(x)dx$$
$$= \int_{-\delta}^{b} g(x)\mathscr{V}\ell_j^*(x)dx,\ j = m,\ m+1,\ \cdots, n. \tag{15}$$

The kernel form of Eq. (15) is given as:

$$\chi\vartheta = \Lambda \tag{16}$$

where χ is $n \times (n+m)$, ϑ and Λ are $n \times 1$ matrices, to have unique equations, we apply m independent conditions given in Eq.(2) to obtain:

$$\sum_{j=0}^{m-1}\sum_{r=1}^{i} \rho_{s,r}{}^{(j)} \left(\frac{4}{b-a}\right)^{j} \alpha_j \frac{d^j}{dx^j}\left(V\ell_i^*(x)\right)\Big|_{x=\tau_r} = \eta_s, \quad s = 1,\ 2, \ldots,\ m. \tag{17}$$

We then solve Eqs. (15) and (17) simultaneously to obtain the unknowns, $j = 0,1,\cdots,n$, and subsequently the approximate solution given by Eq. (13).

4. Numerical Implementation

In this section, we implement the scheme discussed in section [3] on some selected problems in the literature. To demonstrate the accuracy of the scheme, we compute the absolute error and compare with the existing results in the literature.

Example 4.1

Consider the following delay Fredholm IDE (Issa *et al*., 2019); (Shahmorad and Ostadzad, 2016):

$$\begin{cases} \int_0^1 xty\left(t-\frac{1}{2}\right)dt - 2y(x) + \dfrac{dy}{dx} - \left(\frac{1}{4}\exp(1) - \frac{61}{48}\right)x = -1, & x \in [0,\ 1] \\ \\ y(x) = 2, & x \in \left[-\frac{1}{2},\ 0\right] \end{cases} \tag{18}$$

subject to $y(0) = 2$. The exact solution $y(x) = x + e^{2x} + 1$.

From the scheme discussed in section 3, Eq. (18) becomes:

$$\begin{aligned} &\frac{dy_n}{dx} - 2y_n(x) + \int_0^1 xty_n\left(t-\frac{1}{2}\right)dt - \left(\frac{1}{4}\exp(1) - \frac{61}{48}\right)x + 1 + 2\int_0^{\frac{1}{2}} xtdt = 0 \\ &\frac{d}{dx}\left(\sum_{i=0}^{n}\alpha_i \mathcal{V}\ell_i^*(x)\right) - 2\sum_{i=0}^{n}\alpha_i\mathcal{V}\ell_i^*(x) + \int_0^1 xt\sum_{i=0}^{n}\alpha_i\mathcal{V}\ell_i^*\left(t-\frac{1}{2}\right)dt - \left(\frac{1}{4}\exp(1) - \frac{61}{48}\right)x \\ &+ 1 + 2\int_0^{\frac{1}{2}} xtdt = 0. \end{aligned} \tag{19}$$

Multiplying Eq. (19) by the shifted Vieta-Lucas $V\ell_j^*(t), j = 1,2,\ldots,n$ and integrating, we obtain $n \times (n+m)$. To have unique equations, we use the attached condition:

$$y(0) = 2 \Rightarrow \sum_{i=0} \alpha_i\mathcal{V}\ell_i^*(x=0) = 2, \tag{20}$$

We then solve Eqs. (19) and (20) simultaneously to obtain the unknowns and, subsequently, the approximate solutions.

The absolute errors corresponding to $n = 15$ are given in Table 1, and Table 3 displays the comparison of the errors. Figure 1 is the graphical representation of the computational results.

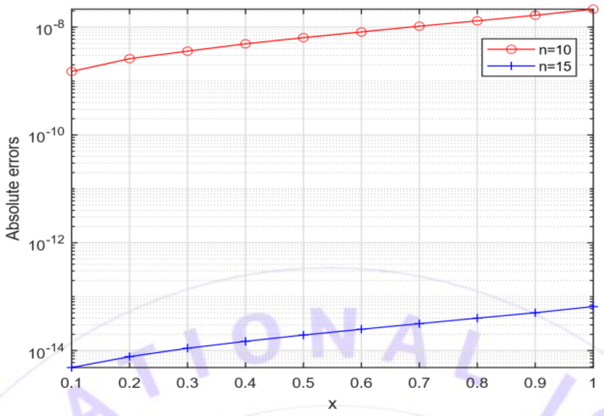

Figure 1: Absolute errors at different values of *n* for Example 4.1

Numerical 4.2

Consider the following third order delay Volterra IDE with variable coefficient (Issa *et al*., 2019); (Shahmorad & Ostadzad, 2016):

$$y(x)+xy'(x)+y''(x)+xy'''(x)+\int_{\frac{1}{2}}^{x} y\left(t-0.5\right)dt$$
$$=\frac{1}{2}\left(x-0.5\right)^2+2\left(x-0.5\right)+2-\cos\left(x-0.5\right),\ x\in(0,\ 1\ ]$$

subject to the attached conditions:

$$y(0)\ =\ 0,\ y(0.5)\ +y'(0.5)\ =\ 1.5\ +\ \sin(0.5)\ +\ \cos(0.5)$$
$$y(1)+y(0)=1+\sin(1)$$

The close form solution is $y(x)=\sin(x)+x$.

Table 1 displays the errors at $n = 15$, and Table 3 exhibits the comparison of the maximum absolute errors relative to the existing results. Figure 2 shows the graphical results.

Table 1: Absolute errors - Examples 4.1 and 4.2 at $n = 15$

x	Example 4.1	Example 4.2
0	0	0
0.1	4.85×10^{-15}	8.26×10^{-18}
0.2	7.72×10^{-15}	1.23×10^{-17}
0.3	1.11×10^{-14}	1.40×10^{-17}

0.4	1.49×10^{-14}	1.42×10^{-17}
0.5	1.94×10^{-14}	1.33×10^{-17}
0.6	2.49×10^{-14}	1.16×10^{-17}
0.7	3.16×10^{-14}	9.32×10^{-18}
0.8	4.00×10^{-14}	6.54×10^{-18}
0.9	5.03×10^{-14}	3.40×10^{-18}
1.0	0	0

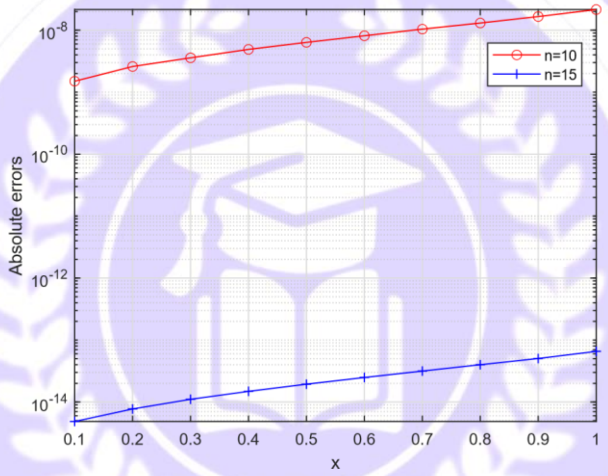

Figure 2: Absolute errors at different values of *n* for Example 4.2

Example 4.3

Consider (Issa (2019), Saadatmandi and Dehghan (2010)):

$$xy(x-1)+xy'(x)-y''(x-1)-y'''(x)+\int_{-1}^{1} y(t-1)dt = (x+1)\left[\sin(x-1)+\cos x\right]+\cos(2)-1 \tag{22}$$

subject to the attached conditions: $y''(0)=y(0)=0, y'(0)=1.$

With the close form solution $y(x) = \sin(x)$.

Table 2 is the comparison of the maximum absolute errors for $n = 6$ and $n = 7$.

Table 2: Comparison of errors for Example 4.3 at n = 6 and n = 7

n	Saadatmandi and Dehghan (2010)	Issa *et al.* (2019)	PS
6	1.82×10^{-2}	1.04×10^{-2}	9.20×10^{-3}
7	5.05×10^{-3}	4.87×10^{-3}	1.48×10^{-3}

Example 4.4

Consider Fredholm IDE (Issa *et al*., 2019); (Yűzbaşi, 2017):

$$y(x) - y'(x) + \int_0^1 e^{tx} y(t)dt - \frac{e^{(1+x)} - 1}{1+x} = 0,\ \ y(0) = 1 \quad , \tag{23}$$

with the exact solution is $y(x) = e^x$.

Table 3 presents the maximum absolute errors relative to the results obtained in (Issa *et al*., 2019); **(**Yűzbaşi, 2017).

Example 4.5

Consider the delay Volterra IDE (Issa *et al*., 2019); (Shahmorad and Ostadzad, 2016):

$$y(x+1) - y'(x) - 4y(x) - 3\int_{x-1}^{x} y(t)dt + 2e^{(1-x)} = 0,\ \ x \geq 0,\ \ y(0) = 1 \quad , \tag{24}$$

with the exact solution is $y(x) = e^{-x}$.

Maximum absolute errors are displayed in Table 3. Figure 3 is the graphical representation corresponding to the computational results at $n = 8$ and $n = 15$.

Table 3: Comparison of errors

Examples	**N**	**Shahmorad and Ostadzad,(2016)**	**Yűzbaşi, (2017)**	**Issa *et al.* (2019)**	**PS**
4.1	10	4.50×10^{-7}	-	2.30×10^{-8}	1.20×10^{-8}
4.2	9	7.50×10^{-3}	-	7.20×10^{-10}	6.80×10^{-11}
4.4	12	-	4.48×10^{-11}	6.71×10^{-17}	1.42×10^{-17}
4.5	8	-	-	8.19×10^{-7}	3.60×10^{-8}

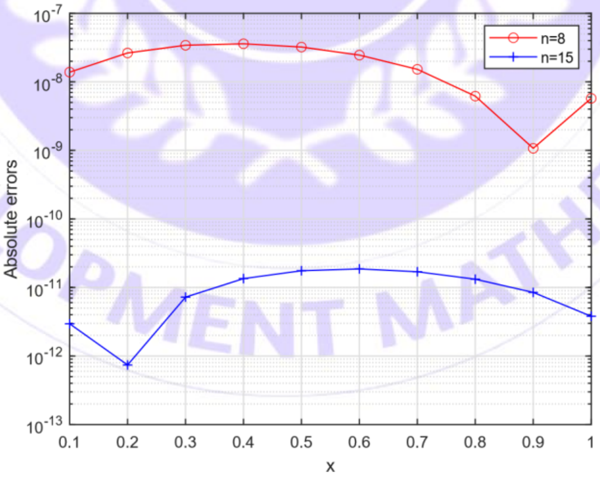

Figure 3: Absolute errors for Example 4.5

4. Results Discussion

Table 1 presents the errors at n = 15, for example, 4.1 and 4.2, while Figures 1 and 2 exhibit the absolute errors corresponding to the first two examples at n = 10 and n = 15, and Figure 3 is the graphical representation of example 4.5 at n = 8 and n = 15. Table 2 presents the maximum errors, for example, 4.3 at n = 6 and n = 7, while Table 3 displays the comparison of the maximum absolute errors relative to the existing results.

5. Conclusions

The techniques proposed in this article employ the Galerkin method to transform the GDIDE into a system of linear algebraic equations using Vieta-Lucas polynomials as an approximating polynomial. The technique was extended to solve some integro-differential-difference equations. The efficiency and accuracy of the scheme were examined on some selected existing examples from the literature. Obviously, results obtained justify the accuracy and effectiveness of the method and it was also observed that, the higher the degree of approximation the better the accuracy.

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Competing Interests

Authors have declared that there is no conflict of interest reported in this work.

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