

Mathematical Analysis of Substance Abuse and its Effect

Ahmed K. Dotia^{a*}, Mohammed O. Ibrahim^b, Adamu Gambo^a, Ahmed B. Musa^b and Abisola O. Lawani^c

^aDepartment of Mathematics, Faculty of Natural and Applied Science, Nigerian Army University Biu, Borno State, Nigeria

^bDepartment of Mathematics, Faculty of Natural Science, University of Ilorin, Ilorin, Kwara State, Nigeria

^cDepartment of Mathematics, Tai Solarin University of Education, Ijebu-Ode, Ogun State, Nigeria

ARTICLE INFO

Article history:

Received 17 February 2024

Received in revised form 05 May 2024

Accepted 19 May 2024

Keywords:

Basic reproduction number,
Mathematical model, Substance abuse,
Population, Stability,

MSC 2020 Subject classification:

92B05, 92-10

ABSTRACT

In this paper, a model for substance abuse was formulated. A compartmentalization approach was adopted to develop the model, and five nonlinear equations were developed based on the number of compartments. The model was analyzed qualitatively and quantitatively. The Invariant region, positivity, equilibrium points (substance-free and substance-present), basic reproduction number R_0 , the sensitivity analysis, and the model's local and global asymptotic stability were obtained. Numerical simulations were conducted to see the changes in the population dynamics of different classes in the sub-population. The results authenticated that the recovery rate of moderate substance users and the recovery rate of rehabilitation/treatment of individuals are indications that they help reduce the number of substance abuse individuals. Also, the recruitment and contact rates impact the population of substance abuse.

1. Introduction

Substance/drug abuse can be viewed as excess intake or dependence on a substance, drug, or another chemical foremost to effects injurious to the individual's physical, social, psychological, and mental health and the welfare of others. Most substances abused are psychoactive, which leads to addiction syndrome when used. Substances abused include legal (not prohibited by law) and criminal (prohibited by law) drugs. They can be classified into different classes depending on the level of abuse and the interest of the classifying organization.

Globally, substance use is associated with substantial morbidity and mortality rates. In the 2017, Global Burden of Disease (GBD) study, substance use disorders (SUDs) were the second highest cause of disability among mental disorders, with 31,052,000 (25%) Years Lived with Disability (YLD) attributed to them (James, S. L., *et al.*, 2018). In 2016, toxic alcohol usage resulted in 3 million casualties (5.3% of all deaths) worldwide and 132.6 (5.1%) million disability-adjusted life years (DALYs) (Hammer, *et al.*, 2018). Alcohol and tobacco use are the ultimate danger for non-communicable ailments, such as cardiovascular disease, cancer, and liver disease (WHO, 2017). Even though the prevalence rate of opioid use is low compared to that of tobacco and alcohol use, opioid use disorder contributes to 76% of all deaths from SUDs (WHO, 2017). Additional psychoactive substances such as Cannabis and amphetamines are associated with cognitive health consequences, including increased risk of suicide, depression, anxiety, and psychosis (McKetin, *et al.*, 2019; Lowe, *et al.*, 2019). In addition to the impact on health, substance use is associated with considerable socio-economic costs stemming from its effect on health and criminal justice systems (Jaguga, *et al.*, 2022).

Drug abuse and addiction are characterized by a self-destructive pattern of substance use that leads to significant problems and distress, including tolerance to or withdrawal from the substance. These drug abuse and addiction are grouped as substance or drug use disorders. Unfortunately, Drug use disorder is quite common, affecting more than 8% of people in the country at some point in their lives. People sometimes abuse virtually any substance whose ingestion can result in a euphoric or high feeling. At the same time, the physical and psychological execution of drug use disorder tends to vary based on the substance involved. The general effects of a substance draw on disorder involving any drug can be devastating (Perez, 2020).

*Corresponding author. Tel.: +2348039638366

E-mail address: akdotia@gmail.com (Dotia A. K.)

<https://doi.org/10.62054/ijdm/0102.06>

In 2018, (Kanyaa, *et al.*, 2018) analyzed four compartmental models that explain the dynamics of commercial drivers' use and abuse of substances. They reveal that an increase in the contact or imitation rate increases the number or population of drug users. (Khajji, *et al.*, 2020) also established and analyzed a continuous mathematical model of Alcohol drinking with the influence of private and public addiction treatment centers in their paper. The basic properties of the model were studied and discussed. However, both (Kanyaa, *et al.*, 2018; Khajji, *et al.*, 2020) only discussed an incidence with the potential drinker or susceptible substance abuse in their model. Relapse of recovery into potential drinkers was not incorporated into their model. This study will discuss the potential substance abuse, rehabilitation/treatment, and moderate, recovered, and heavy substance abuse classes. Some parameters were added to have a robust mathematical model of substance abuse among individuals in society.

1. Substance model description and formulation

The substance abuse model is divided into five compartments depending on their status. We have the potential substance abuse $P(t)$, moderate substance abuse $M(t)$, heavy substance abuse $H(t)$, rehabilitation/treatment $Q(t)$, and quitters / removed $R(t)$. The graphical representation of the proposed model is shown in Figure 1 below.

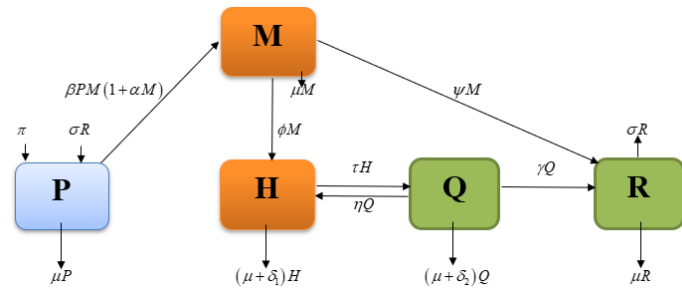


Figure 1: Flow diagram of Substance Abuse transmission

We consider the following system of five nonlinear differential equations:

$$\left. \begin{aligned}
 \frac{dP}{dt} &= \pi + \sigma R - \beta P M (1 + \alpha M) - \mu P \\
 \frac{dM}{dt} &= \beta P M (1 + \alpha M) - (\phi + \psi + \mu) M \\
 \frac{dH}{dt} &= \phi M + \eta Q - (\tau + \delta_1 + \mu) H \\
 \frac{dQ}{dt} &= \tau H - (\eta + \gamma + \delta_2 + \mu) Q \\
 \frac{dR}{dt} &= \gamma Q + \psi M - (\sigma + \mu) R
 \end{aligned} \right\} \quad (1)$$

1.1. Model Assumptions

The model formulated is based on the assumptions below:

- i. an individual in subgroup P is enrolled at a constant rate.
- ii. the same natural death rate is chosen for all individuals involved in this study.
- iii. all individuals with potential substance abuse are equally likely to become substance users.

- iv. on recovery, an individual becomes temporarily invulnerable to substance abuse.
- v. all individuals are born with potential substance abuse.
- vi. only heavy drug, alcohol, and substance users may die from addiction.
- vii. potential individuals become substance users through social contact and imitation.
- viii. it is assumed that there is no interaction between the susceptible population and heavy drug abuse.

The total population size at time t is denoted by $N(t)$ and can be obtained from the above model.

$$N(t) = P(t) + M(t) + H(t) + Q(t) + R(t) \quad (2)$$

Table 1: Description of the State Variables and Parameters used

Variables		Descriptions	
$P(t)$		potential substance abuse class at the time t	
$M(t)$		moderate substance abuse class at the time t	
$H(t)$		heavy substance abuse class at the time t	
$Q(t)$		rehabilitation/treatment class at the time t	
$R(t)$		quitters / removed class at the time t	
Parameters	Interpretation	Values	References
π	the rate at which individuals are recruited into the Potential abuse class	65	(Khajji, <i>et al.</i> , 2020)
μ	Natural mortality rate	0.019-0.021	(Bradshaw, and Timaeus, 2006)
δ_1	heavy substance abuse-related mortality rate	0.0015-0.0035	Estimated
δ_2	rehabilitation/treatment-related mortality rate	0.0013-0.002	Estimated
β	Transmission coefficient or effective contact rate on moderate abuse	0.01-0.08	(Khajji, <i>et al.</i> , 2020)
η	relapse of rehabilitated individuals to heavy abuse	0.001-0.005	Estimated
σ	Rate of movement from recovered to potential abuse	0.001-0.007	Estimated
τ	Rate of movement from heavy abuse to rehabilitation /treatment	0.001-0.009	(Khajji, <i>et al.</i> , 2020)
γ	Rate of movement from rehabilitation/treatment to recovered class	0.001-0.009	(Khajji, <i>et al.</i> , 2020)

ϕ	Rate of movement from moderate abuse to heavy abuse	0.1-0.9	(Santoro, <i>et al.</i> , 2013)
ψ	Rate of movement from moderate abuse to recovered /removed	0.001-0.009	Estimated
α	the rate at which individual imitate their colleagues who use a moderate substance	0.01-0.05	Estimated

3.0 Basic Properties

3.1 Invariant Region

It is necessary to prove that all solutions of system (1) above with positive initial data will remain positive for all times $t > 0$. The following lemma will establish this.

Lemma 1. All feasible solution $P(t)$, $M(t)$, $H(t)$, $Q(t)$, and $R(t)$ of system equation (1) is bounded by the region: $\Omega = (P, M, H, Q, R) \in \mathfrak{R}_+^5$

Proof. From the system equation(1)

$$\frac{dN}{dt} = \frac{dP}{dt} + \frac{dM}{dt} + \frac{dH}{dt} + \frac{dQ}{dt} + \frac{dR}{dt} \quad (3)$$

$$\frac{dN}{dt} = \pi - \delta_1 H - \delta_2 Q - \mu N \quad (4)$$

Here, it is important to note that in the absence of the substance,

$$\frac{dN(t)}{dt} \leq \pi - \mu N \quad (5)$$

and it follows that

$$N(t) \leq \frac{\pi}{\mu} - \frac{N(0)}{\mu} e^{-\mu t} \quad (6)$$

where $N(0)$ is the initial population; thus,

$$\limsup_{t \rightarrow \infty} N(t) \leq \frac{\pi}{\mu} \quad (7)$$

hence,

$$P(t) + M(t) + H(t) + Q(t) + R(t) \leq \frac{\pi}{\mu} \quad (8)$$

Hence, for model (1) analysis, we get the region given by the set.

$$\Omega = (P, M, H, Q, R) \in \mathfrak{R}_+^5 : P + M + H + Q + R \leq \frac{\pi}{\mu} \quad (9)$$

which is a positively invariant set. Hence the system is both mathematically and epidemiologically well posed -. Thus we restrict our analysis to the region Ω . (where the models make biological sense)

3.2 Positivity of Solutions of the Model

Theorem 1. If $P(0) > 0$, $M(0) \geq 0$, $H(0) \geq 0$, $Q(0) \geq 0$, and $R(0) \geq 0$, then the solution of system equation (1) $P(t)$, $M(t)$, $H(t)$, $Q(t)$, and $R(t)$ are positive for all $t > 0$.

Proof. From the first equation of system (1), we have;

$$\frac{dP}{dt} \geq \pi - [BM(1 + \alpha M) + \mu]P \quad (10)$$

assumed that $G(t) = BM(1 + \alpha M) + \mu$, (11)

multiply equation (10) by $\exp\left(\int_0^t G(s) ds\right)$ and we obtain

$$\frac{dP}{dt} \times \exp\left(\int_0^t G(s) ds\right) \geq [\pi - G(t)P] \times \exp\left(\int_0^t G(s) ds\right) \quad (12)$$

the above implies

$$\frac{dP}{dt} \times \exp\left(\int_0^t G(s) ds\right) + G(t)P \times \exp\left(\int_0^t G(s) ds\right) \geq \pi \times \exp\left(\int_0^t G(s) ds\right) \quad (13)$$

Thus

$$\frac{d}{dt} \left[P \times \exp\left(\int_0^t G(s) ds\right) \right] \geq \pi \times \exp\left(\int_0^t G(s) ds\right) \quad (14)$$

integral with respect to s from 0 to t, we obtain

$$P(t) \times \exp\left(\int_0^t G(s) ds\right) - P(0) \geq \pi \times \int_0^t \left(\exp\left(\int_0^s G(s) ds\right) \right) dw \quad (15)$$

Multiplying equation (14) by $\left(-\int_0^t G(s) ds\right)$, we have

$$P(t) - P(0) \times \exp\left(-\int_0^t G(s) ds\right) \geq \pi \times \exp\left(-\int_0^t G(s) ds\right) \times \int_0^t \left(\exp\left(\int_0^s G(s) ds\right) \right) dw \quad (16)$$

Thus,

$$P(t) \geq P(0) \times \exp\left(-\int_0^t G(s) ds\right) + \pi \times \exp\left(-\int_0^t G(s) ds\right) \times \int_0^t \left(\exp\left(\int_0^s G(s) ds\right) \right) dw \geq 0 \quad (17)$$

So, the solution $P(t)$ is positive.

Similarly, for the other equations of system (1), we have;

$$M(t) \geq M(0) \times \exp\left(\int_0^t G_m(s) ds\right) \geq 0 \quad (18)$$

Where

$$G_m = \beta P(1 + \alpha M) - (\phi + \psi + \mu) \quad (19)$$

hence,

$$H(t) \geq H(0) \times \exp(-(\tau + \delta_1 + \mu)t) \geq 0 \quad (20)$$

$$Q(t) \geq Q(0) \times \exp(-(\eta + \delta_2 + \mu)t) \geq 0 \quad (21)$$

$$R(t) \geq R(0) \times \exp(-(\sigma + \mu)t) \geq 0 \quad (22)$$

Therefore, we observe from the above that $P(t) > 0$, $M(t) > 0$, $H(t) > 0$, $Q(t) > 0$, and

$R(t) > 0 \forall t \geq 0$, and this completes the proof.

3.1 Equilibria and Stability Analysis

3.1.1 Equilibrium Point

In this model, there are two equilibrium points: the substance-free and the substance-present equilibrium points. The equilibrium points are found by setting the right-hand side of equation (1) equal to zero.

The substance-free equilibrium $E^0\left(\frac{\pi}{\mu}, 0, 0, 0, 0\right)$ is achieved in the absence of substance abuse ($M = H = Q = R = 0$).

The substance-present equilibrium $E^*(P^*, M^*, H^*, Q^*, R^*)$ is achieved when substance exist ($M \neq 0$ and $H \neq 0$), where

$$\left. \begin{aligned} P^* &= \alpha\beta\mu + \pi\alpha\beta + A_1^2 - A_1\beta \\ M^* &= \frac{\beta(\pi\alpha\beta - \alpha\beta\mu - A_1^2 + A_1\beta) + A_1}{(-\pi\alpha\beta + \alpha\beta\mu + A_1^2 - A_1\beta)\alpha\beta} \\ H^* &= \frac{A_3\phi(\beta(\pi\alpha\beta - \alpha\beta\mu - A_1^2 + A_1\beta) + A_1)}{(-\pi\alpha\beta + \alpha\beta\mu + A_1^2 - A_1\beta)\alpha\beta(A_2A_3 - \eta\tau)} \\ Q^* &= \frac{\phi\tau(-\beta(-\pi\alpha\beta + \alpha\beta\mu + A_1^2 - A_1\beta) + A_1)}{(-\pi\alpha\beta + \alpha\beta\mu + A_1^2 - A_1\beta)\alpha\beta(A_2A_3 - \eta\tau)} \\ R^* &= \frac{((\pi\alpha\beta - \alpha\beta\mu - A_1^2 + A_1\beta)\mu + \pi)(A_2A_3\psi - \eta\psi\tau + \gamma\phi\tau)}{A_1(A_2A_3 - \eta\tau)A_4} \end{aligned} \right\} \quad (23)$$

where, $A_1 = (\phi + \psi + \mu)$, $A_2 = (\tau + \delta_1 + \mu)$, $A_3 = (\eta + \gamma + \delta_2 + \mu)$, $A_4 = (\sigma + \mu)$

3.2 Basic Reproduction Number

R_0 is the basic reproduction number that measures the average number of new substance abusers generated by a single substance abuser in a population of potential substance abusers. The R_0 value will indicate whether the epidemic could occur or persist. The basic reproduction number can be determined by using the next-

generation matrix method. The following basic reproduction number was obtained;

$$R_0 = \frac{\beta\pi}{\mu(\phi + \psi + \mu)} \quad (24)$$

3.3 Local Stability Analysis

Now we proceed to study the stability behavior of equilibria E^0 and E^* .

3.3.1 The Substance-Free Equilibrium.

In this section, we analyse the local stability of the substance-free equilibrium.

Theorem 2. The substance-free equilibrium $E^0 \left(\frac{\pi}{\mu}, 0, 0, 0, 0 \right)$ of the system (1) is asymptotically stable if

$R_0 < 1$ and unstable if $R_0 > 1$.

Proof. The Jacobian matrix at E is given by

$$J(E) = \begin{pmatrix} -K_1 - \mu & -K_2 & 0 & 0 & \sigma \\ K_1 & K_2 - A_1 & 0 & 0 & 0 \\ 0 & \phi & -A_2 & \eta & 0 \\ 0 & 0 & \tau & -A_3 & 0 \\ 0 & \psi & 0 & \gamma & -A_4 \end{pmatrix} \quad (25)$$

where, $K_1 = \beta M(1 + \alpha M)$, $K_2 = \beta M(1 + \alpha M) + \alpha \beta M P$, $A_1 = (\phi + \psi + \mu)$, $A_2 = (\tau + \delta_1 + \mu)$,

$A_3 = (\eta + \gamma + \delta_2 + \mu)$, $A_4 = (\sigma + \mu)$

The Jacobian matrix for the substance-free equilibrium is given by

$$J(E^0) = \begin{pmatrix} -\mu & -\frac{\pi\beta}{\mu} & 0 & 0 & \sigma \\ 0 & \frac{\pi\beta}{\mu} - A_1 & \tau & -A_3 & 0 \\ 0 & \phi & -A_2 & \eta & 0 \\ 0 & 0 & \tau & -A_3 & 0 \\ 0 & \psi & 0 & \gamma & -A_4 \end{pmatrix} \quad (26)$$

where $P^0 = \frac{\pi}{\mu}$, The characteristic equation of this matrix is given by $\det(J(E^0) - \lambda I_5) = 0$ where I_5 is a square identity matrix of order 5.

Therefore, eigenvalues of the characteristic equation of $J(E^0)$ are

$$\left. \begin{aligned} \lambda_1 &= -\mu \\ \lambda_2 &= -A_4 \\ \lambda_3 &= -\frac{A_2 + A_3 - \sqrt{c_1}}{2} \\ \lambda_4 &= -\frac{A_2 + A_3 + \sqrt{c_1}}{2} \\ \lambda_5 &= -\frac{A_1\mu - \pi\beta}{\mu} \end{aligned} \right\} \quad (27)$$

$$\text{where } c_1 = A_2^2 - 2A_2A_3 + A_3^2 + 4\eta\tau$$

Therefore, all the eigenvalues of the characteristic equation $J(E^0)$ are real and negative if $R_0 < 1$.

We conclude that the substance-free equilibrium is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

3.2.1 Substance-Present Equilibrium

Here, we analyse the local stability of the substance-present equilibrium.

Let the following theorem analyze the substance's local stability present equilibrium when $R_0 > 1$.

Theorem 3. The substance present equilibrium E^* is locally asymptotically stable if $R_0 > 1$ and unstable otherwise.

Proof. We present $E^*(P^*, M^*, H^*, Q^*, R^*)$ substance present equilibrium of system (1) and $P \neq 0$, $M \neq 0$, $H \neq 0$, $Q \neq 0$ and $R \neq 0$.

The Jacobian matrix is

$$J(E^*) = \begin{pmatrix} -K_1 - \mu & -K_2 & 0 & 0 & \sigma \\ K_1 & K_2 - A_1 & 0 & 0 & 0 \\ 0 & \phi & -A_2 & \eta & 0 \\ 0 & 0 & \tau & -A_3 & 0 \\ 0 & \psi & 0 & \gamma & -A_4 \end{pmatrix} \quad (28)$$

where, $K_1 = \beta M(1 + \alpha M)$, $K_2 = \beta M(1 + \alpha M) + \alpha\beta MP$, $A_1 = (\phi + \psi + \mu)$, $A_2 = (\tau + \delta_1 + \mu)$,

$A_3 = (\eta + \gamma + \delta_2 + \mu)$, $A_4 = (\sigma + \mu)$

We see that the characteristic equation of the above system is $P(\lambda)$ of $J(E^*)$ the eigenvalues are $\lambda_1 = -(K_1 + \mu)$, $\lambda_2 = -A_3$, $\lambda_3 = -A_4$ whose real part is negative. So, in order to determine the stability of the substance

present equilibrium of model (1), we shall discuss the remaining roots, which constitute the following equation

$$X(\lambda) = \lambda^2 + a_1\lambda + a_2 \quad (29)$$

where,

$$\begin{aligned} a_1 &= \tau + \delta_1 + \mu + \phi + \psi + \mu + \beta P(1 + \alpha M) - \alpha \beta PM > 0 \\ a_2 &= (\beta P(1 + \alpha M)) - \alpha \beta PM - (\tau + \delta_1 + \mu)(\phi + \psi + \mu) > 0 \end{aligned} \quad (30)$$

using Routh–Hurwitz criterion, the system (1) is locally asymptotically stable if $a_1 > 0$ and $a_2 > 0$. Therefore, the present substance equilibrium E^* system is locally asymptotically stable.

3.3 Global Stability

3.3.1 Global Stability of the Substance-Free Equilibrium.

We use the Lyapunov function theory for both the substance-free and the substance-present equilibrium to show that system (1) is globally asymptotically stable. First, we present the global stability of the substance-free equilibrium E^0 when $R_0 \leq 1$.

Theorem 4. The substance-free equilibrium E^0 of the model (1) is globally asymptotically stable (GAS) in ω if $R_0 \leq 1$ and unstable otherwise.

Proof.

To establish the global stability of the substance -free equilibrium, the two conditions of Castillo-Chavex, (2002) must be satisfied for $R_0 < 1$. The model is rewritten in the form:

$$\frac{dX_1}{dt} = F(X_1, X_2), \quad \frac{dX_2}{dt} = G(X_1, X_2): G(X_1, X_2) = 0 \quad (31)$$

where $X_1 = (P, Q)$ and $X_2 = (M, H)$. Since the substance-free is

$$E_f = (X_1^*, 0), \quad X_1^* = \left(\frac{\pi}{\mu}, 0 \right) \quad (32)$$

First condition, that is globally asymptotically stability of X_1^* , we have

$$\frac{dX_1}{dt} = F(X_1, 0) = \begin{pmatrix} \pi + \sigma R - \mu P \\ 0 \end{pmatrix} \quad (33)$$

then the behavior of the computation can be obtained by solving the system of ordinary differential equation, assuming $\pi = \mu$ and hence we get

$$\begin{pmatrix} P \\ R \end{pmatrix} = \begin{pmatrix} 1 + P(0)e^{-\mu t} \\ R(0)e^{-(\sigma + \mu)t} \end{pmatrix} \quad (34)$$

Now since $\lim_{t \rightarrow \infty} X^0$ as $t \rightarrow \infty$ Thus, $X_1^* = \left(\frac{\pi}{\mu}, 0 \right)$ is globally asymptotically stable. For the Second condition

$$G(X_1, X_2) = AX_2 - G(X_1, X_2) \quad (35)$$

$$AX_2 = \begin{pmatrix} \beta P - A_1 & 0 \\ \phi & -A_2 \end{pmatrix} \begin{pmatrix} M \\ H \end{pmatrix} \quad (36)$$

$$G(X_1, X_2) = \begin{pmatrix} (\beta P - A_1)M - A_1 M \\ \phi M + \eta Q - A_2 H \end{pmatrix} \quad (37)$$

$$G(X_1, X_2) = AX_2 - G(X_1, X_2) = \begin{pmatrix} A_1 M \\ \eta Q \end{pmatrix} \quad (38)$$

Its obvious that since $\bar{G}(X_1, X_2) \geq 0$ where $0 \leq (M, H, Q) \leq 1$ thus the proof is complete, hence E_0 is globally asymptotically stable.

3.3.2 Global Stability of the Substance-present Equilibrium.

The result of the global stability of E^* in this section is as follows.

Theorem 5. If $R_0 > 1$, then the substance-endemic equilibrium E^* , of the model (1), given by (23) is globally asymptotically stable in the interior of the region Ω .

Proof. Consider the Lyapunov function $V: \Gamma \rightarrow \mathbb{R}$

$$V(P, M, H, Q, R) = c_1 \left(P - P^* - P^* \ln \left(\frac{P}{P^*} \right) \right) + c_2 \left(M - M^* - M^* \ln \left(\frac{M}{M^*} \right) \right) \\ + c_3 \left(H - H^* - H^* \ln \left(\frac{H}{H^*} \right) \right) + c_4 \left(Q - Q^* - Q^* \ln \left(\frac{Q}{Q^*} \right) \right) + c_5 \left(R - R^* - R^* \ln \left(\frac{R}{R^*} \right) \right) \quad (39)$$

Where, $c_1 = 1$, $c_2 = \frac{A_2}{\phi}$, $c_3 = \frac{A_3}{\tau}$, $c_4 = \frac{A_4}{\sigma}$, and $c_5 = \frac{A_4}{\sigma}$ are positive constants and $\Gamma = [(P, M$

$) \in \Gamma / P > 0, M > 0]$. Then, the derivative of the Lyapunov function is given by

$$\frac{dV(P, M, H, Q, R)}{dt} = P' - \frac{P^*}{P} P' + \frac{A_1}{\phi} \left(M' - \frac{M^*}{M} M' \right) + \frac{A_2}{\tau} \left(H' - \frac{H^*}{H} H' \right) + \frac{A_3}{\gamma} \left(Q' - \frac{Q^*}{Q} Q' \right) \\ + \frac{A_4}{\sigma} \left(R' - \frac{R^*}{R} R' \right) \quad (40)$$

$$\begin{aligned} \frac{dV(P, M, H, Q, R)}{dt} = & \left(\pi + \sigma R - \beta PM(1 + \alpha M) - \mu P - \frac{P^*}{P} (\pi + \sigma R - \beta PM(1 + \alpha M) - \mu P) \right) \\ & + \frac{A_1}{\phi} \left(\beta PM(1 + \alpha M) - (\phi + \psi + \mu)M - \frac{M^*}{M} (\beta PM(1 + \alpha M) - (\phi + \psi + \mu)M) \right) \\ & + \frac{A_2}{\tau} \left(\phi M + \eta Q - (\tau + \delta_1 + \mu)H - \frac{H^*}{H} (\phi M + \eta Q - (\tau + \delta_1 + \mu)H) \right) \\ & + \frac{A_3}{\gamma} \left(\tau H - (\eta + \gamma + \delta_2 + \mu)Q - \frac{Q^*}{Q} (\tau H - (\eta + \gamma + \delta_2 + \mu)Q) \right) \\ & + \frac{A_4}{\sigma} \left(\gamma Q + \psi M - (\sigma + \mu)R - \frac{R^*}{R} (\gamma Q + \psi M - (\sigma + \mu)R) \right) \end{aligned} \quad (41)$$

$$\begin{aligned} \frac{dV(P, M, H, Q, R)}{dt} = & \pi + \sigma R + \beta P^* M(1 + \alpha M) + \mu P^* - \beta PM(1 + \alpha M) - \mu P - \frac{P^*}{P} (\pi + \sigma R) \\ & + \frac{A_1(\beta PM(1 + \alpha M))}{\phi} - \frac{A_1^2 M}{\phi} - \frac{A_1(\beta PM^*(1 + \alpha M))}{\phi} - \frac{A_1^2 M^*}{\phi} + \frac{A_2 \phi M}{\tau} + \frac{A_2 \eta Q}{\tau} \\ & + \frac{A_2^2 H^*}{\tau} - \frac{A_2^2 (\tau + \delta_1 + \mu)H}{\tau} - \frac{A_2 \phi MH^*}{\tau H} - \frac{A_2 \eta QH^*}{\tau H} + \frac{A_3 \tau H}{\gamma} - \frac{A_3^2 Q}{\gamma} - \frac{A_3 Q^* \tau H}{\gamma Q} \\ & + \frac{A_3^2 Q^*}{\gamma} + \frac{A_4 \gamma Q}{\sigma} + \frac{A_4 \psi M}{\sigma} + \frac{A_4^2 R^*}{\sigma} - \frac{A_4^2 R}{\sigma} - \frac{A_4 \gamma QR^*}{\sigma R} - \frac{A_4 \psi MR^*}{\sigma R} \end{aligned} \quad (42)$$

$$\frac{dV}{dt} = L - M \quad (43)$$

where M are the positive terms, and N are the negative terms;

$$\begin{aligned} M = & \pi + \sigma R + \beta P^* M(1 + \alpha M) + \mu P^* + \frac{A_1(\beta PM(1 + \alpha M))}{\phi} + \frac{A_2 \phi M}{\tau} \\ & + \frac{A_2 \eta Q}{\tau} + \frac{A_2^2 H^*}{\tau} + \frac{A_3 \tau H}{\gamma} + \frac{A_3^2 Q^*}{\gamma} + \frac{A_4 \gamma Q}{\sigma} + \frac{A_4 \psi M}{\sigma} + \frac{A_4^2 R^*}{\sigma} \end{aligned} \quad (44)$$

$$\begin{aligned} N = & -\beta PM(1 + \alpha M) - \mu P - \frac{P^*}{P} (\pi + \sigma R) - \frac{A_1^2 M}{\phi} - \frac{A_1(\beta PM^*(1 + \alpha M))}{\phi} - \frac{A_2^2 (\tau + \delta_1 + \mu)H}{\tau} \\ & - \frac{A_1^2 M^*}{\phi} - \frac{A_2 \phi MH^*}{\tau H} - \frac{A_2 \eta QH^*}{\tau H} - \frac{A_3^2 Q}{\gamma} - \frac{A_3 Q^* \tau H}{\gamma Q} - \frac{A_4^2 R}{\sigma} - \frac{A_4 \gamma QR^*}{\sigma R} - \frac{A_4 \psi MR^*}{\sigma R} \end{aligned} \quad (45)$$

If $M < N$ then $\frac{dL}{dt} \leq 0$ and $\frac{dL}{dt} = 0$ if and only $P = P^*, M = M^*, H = H^*, Q = Q^*$, and $R = R^*$. The

largest invariant region $P, M, H, Q, R \in \Omega: \frac{dL}{dt} = 0$. Since all the model parameters are non-negative, the endemic equilibrium is globally asymptotically stable in the invariant Ω . if $M < N$ by La Salle, (1976). Hence V is a Lyapunov function on Ω .

3.7 Sensitivity Analysis of Basic Reproduction Number (R_0)

Sensitivity analysis is used to determine the model's fitness to parameter values, that is, to help us know the parameters that have a high impact on the reproduction number R_0 . Using the approach in Chitnis, *et al.* (2008), we calculate the normalized forward sensitivity indices of R_0 . Let

$$\Upsilon_m^{R_0} = \frac{m}{R_0} \times \frac{\partial R_0}{\partial m} \quad (46)$$

denote the sensitivity index of R_0 with respect to the parameter m . We obtain the results in Tale 3.

Table 3: Sensitivity analysis results

Parameters of R_0	Description	Sensitivity Index
π	recruitment rate into potential substance class	+1.00
β	effective contact rate on moderate substance class	+1.00
μ	natural mortality rate	-1.07
ϕ	movement rate from moderate to heavy substance abuse	-0.21
ψ	movement rate from moderate substance to recovered	-0.73

Table 3 shows that the basic reproduction numbers R_0 are most sensitive to changes in both π and β . If β or π increases R_0 will increase significantly. However, an increase in the parameters μ, ϕ , and ψ will decrease the R_0 , they all have an inversely proportional relationship with R_0 's. So, an increase in any of them will decrease R_0 . However, the size of the decrease will be proportionally smaller. Given R_0 's sensitivity to β and π , it seems sensible to focus on reducing those parameters. In other words, this sensitivity analysis shows us that prevention is better than cure and avoiding an unnecessary association with lousy influence can be beneficial. Efforts channeled towards prevention are more effective in controlling the spread of habitual substance abuse than efforts to increase the number of individuals accessing rehabilitation/treatment.

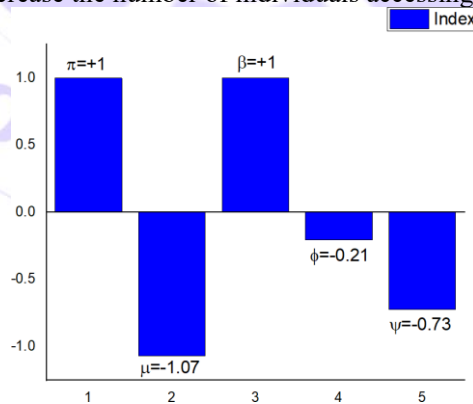
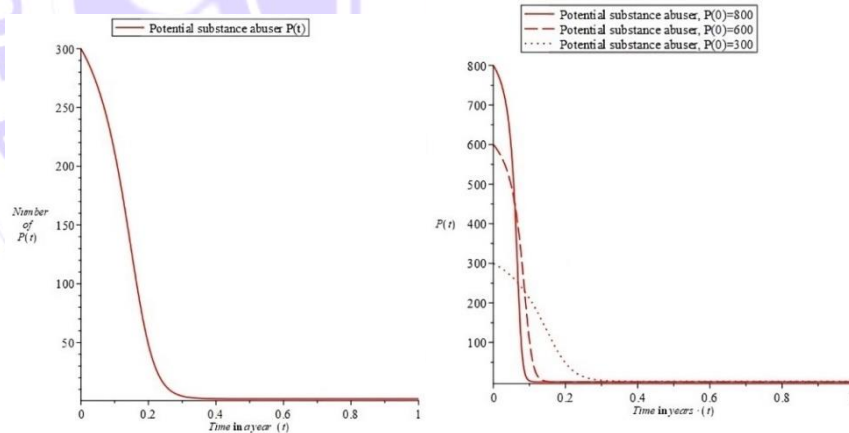


Figure 2: Sensitivity Analysis of Basic Reproduction number

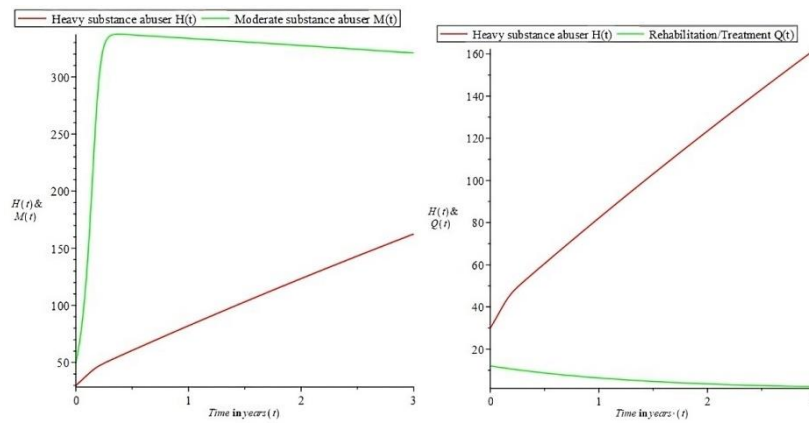
4. Numerical Simulations

In this section, we illustrate some numerical solutions of model (1) for different values of the parameters. here, we use the following different initial values $P(0) = 600$, $M(0) = 50$, $H(0) = 30$, $Q(0) = 0$ and $R(0) = 0$. We use the parameters in Table 1. Some graphic representation of the model was illustrated in the figures. From these figures, using the different values of initial variables $P(t) + M(t) + H(t) + Q(t)$ and $R(t)$, we obtained the following remarks:

1. The number of potential substances decreases drastically in the first year, and the population at various values approaches the number zero (see Figures 3 (a) and (b)).
2. The number of heavy substance abuse increases gradually while the population of moderate substance abuse increases drastically in the first year, and after a peak at 340, it witnesses a slow decline in its progress (see Figure 3 (c))
3. The number of heavy substance abuse increases gradually while the population of rehabilitation or treatment decreases in the first three years (see Figure 3 (d))
4. The number of recovered increases gradually while the population of rehabilitation or treatment decreases in the first five years and increases gradually with time (see Figure 3 (e))
5. The dynamics of recovered, treatment, moderate and heavy substance abuse with time was shown (see Figure 3 (f))
6. The number of moderate substance abuse increases in the first month and later decreases at different population sizes and effective contact rates respectively (see Figures 4 (a) and (b)). Moderate substance abuse increases drastically in the first month but later moves at a constant rate over the years (see Figure 4 (c)).
7. The number of heavy substance abuse increases drastically in the first year but later increases gradually at different effective contact rates (see Figure 4 (d)). Also, heavy substance abuse increases gradually at different population sizes (see Figure 4 (e))
8. The number of rehabilitation/treatments decreases in the first five years and increases afterwards. (see Figure 4 (f))



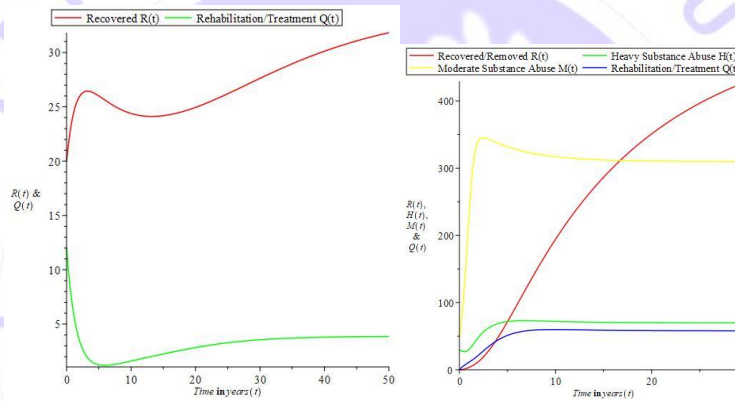
(a) Dynamics of potential substance abuse (b) Dynamics of the population of Potential substance abuse over a year



(c) relationship between Heavy and moderate

(d) Dynamics of Heavy substance abuse and abuse treatment class

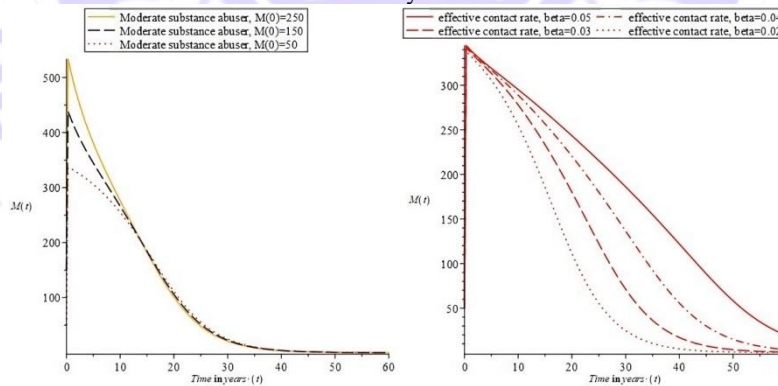
substance



(e) The dynamics of recovered and

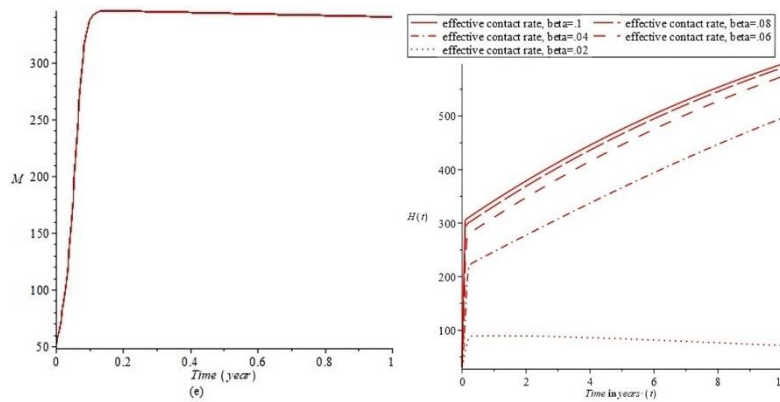
(f) Dynamics of Recovered, treatment, rehabilitation/treatment, moderate and heavy substance abuse

rehabilitation/treatment

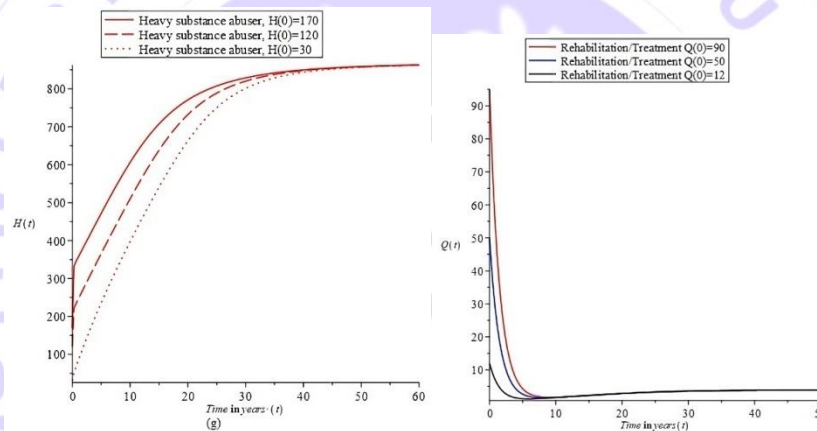


(a) varying population size on moderate

(b) Effect of contact rate on moderate substance abuse



(c) Graph showing moderate substance abuse (d) Effect of contact rate on heavy over one year substance abuse



(e) Graph of Heavy substance abuse over (f) The graph showing the treatment class sixty years. decreases in the first five years.

5. Conclusion

The substance model was qualitatively and quantitatively analysed to account for a substance spread in society. Since substance abuse by nature causes serious health complications, damaging nearly every organ and system in the body. This paper is devoted to implementing a substance abuse mathematical model by introducing rehabilitation/treatment classes and the recovery of moderate substance abuse. The use and abuse of substances will continue to spread among individuals if the reproduction number of the substance abuse is more significant than one. However, if the reproduction number of substance abusers is less than one, the menace would die out of society. Here, the reproduction number was calculated. Sensitivity analysis was therefore carried out to determine the most sensitive parameter on the reproduction number. It would enable us to decide on the factors that lead to the disappearance or spread of substance abuse in society. This analysis of R_0 shows that β , and π are among the most useful parameters to the basic reproduction number. The implication is that the more we allow potential and moderate substance abuse to be in continuous contact, the use and abuse of substances will increase. Moreover, this would lead to the continuous use of the substance by individuals in society. However,

the sign of the values of the recovery rate of moderate substance users and the recovery rate of rehabilitation/treatment of individuals are indications that they help reduce the number of the substance abuse individuals. This might help curb substance abuse by individuals in society.

References

- Bradshaw, D., and Timaeus, I. M. (2006). Levels and trends of adult mortality. *Disease and Mortality in Sub-Saharan Africa. 2nd edition.*
- Castillo-Chavez, C. (2002). On the computation of R. and its role on global stability Carlos Castillo-Chavez*, Zhilan Feng, and Wenzhang Huang. *Mathematical approaches for emerging and reemerging infectious diseases: an introduction*, 1, 229.
- Chitnis, N., Hyman, J. M., and Cushing, J. M. (2008). Determining important parameters in the spread of malaria through the sensitivity analysis of a mathematical model. *Bulletin of Mathematical Biology*, 70, 1272-1296.
- Hammer, J. H., Parent, M. C., and Spiker, D. A. (2018). Mental Help Seeking Attitudes Scale (MHSAS): Development, reliability, validity, and comparison with the ATSPPH-SF and IASMHS-PO. *Journal of Counseling Psychology*, 65(1), 74.
- Jaguga, F., Kiburi, S. K., Temet, E., Barasa, J., Karanja, S., Kinyua, L., and Kwobah, E. K. (2022). A systematic review of substance use and substance use disorder research in Kenya. *PloS one*, 17(6), e0269340.
- James, S. L., Abate, D., Abate, K. H., Abay, S. M., Abbafati, C., Abbasi, N., ... and Briggs, A. M. (2018). Global, regional, and national incidence, prevalence, and years lived with disability for 354 diseases and injuries for 195 countries and territories, 1990–2017: A systematic analysis for the Global Burden of Disease Study 2017. *The Lancet*, 392(10159), 1789-1858.
- Kanyaa, J. K., Osman, S., and Wainaina, M. (2018). Mathematical Modelling of Substance Abuse by Commercial Drivers. *Global Journal of Pure and Applied Mathematics*, 14(9), 1149-1165.
- Khajji, B., Labzai, A., Balatif, O., and Rachik, M. (2020) Mathematical Modeling and Analysis of an Alcohol Drinking Model with the Influence of Alcohol Treatment Centers. *International Journal of Mathematics and Mathematical Sciences*, 2020, 1-12.
- La Salle, J. P. (1976). *The stability of dynamical systems*. Society for Industrial and Applied Mathematics. SIAM, Philadelphia;21
- Lowe, D. J., Sasiadek, J. D., Coles, A. S., and George, T. P. (2019). Cannabis and mental illness: a review. *European archives of psychiatry and clinical neuroscience*, 269(1), 107-120.
- McKetin, R., Leung, J., Stockings, E., Huo, Y., Foulds, J., Lappin, J. M., ... and Degenhardt, L. (2019). Mental health outcomes associated with the use of amphetamines: A systematic review and meta-analysis. *EClinicalMedicine*, 16, 81-97.
- Pérez, E. (2020). Mathematical modeling of the spread of alcoholism among Colombian college students. *Ingeniería y Ciencia*, 16(32), 195-223.

Santoro, M., Triolo, L., and Rossi, C. (2013). Drug user dynamics: A compartmental model of drug users for scenario analyses. *Drugs: education, prevention and policy*, 20(3), 184-194.

World Health Organization. *Alcohol and drug use disorders: Global health estimates. 2017* <http://www.who.int/substanceabuse/activities/fadab/msbadab2017GHE23June2017.pdf>. Accessed 12 June 2021

