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A Hybrid of Direct Computational and Homotopy Analysis Methods for Solving Volterra-Fredholm Integro-Differential Equations

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ABSTRACT

This research considered the problem of Volterra-Fredholm integro-differential equations. A method of Direct computation and Homotopy analysis for solving Volterra-Fredholm integro-differential equations (DHAMVFIDE) was proposed. Convergence analysis to the exact solution of the proposed method was estabished. Examples were solved and comparisons were made with some existing methods to get the efficiency of the proposed method.

1. Introduction

Integro-differential equation (IDE) was first introduced by Volterra in 1900, in connection with the problem of determining population growth. Volterra was studying population growth with focus on hereditary influences, his research work resulted to a problem where both differential and integral operators appeared in one equation. This type of equation is named Volterra integro-differential equation (Wazwaz, 2011). IDEs like IEs play important role in many areas of science such as Mathematics, Physics, Biology, Chemistry and so on. Many engineering problems are formulated by functional equations in the form of IEs, IDEs and differential equations (DE) (Wazwaz, 2011; Kanwal, 1971). Finding solutions of some linear and nonlinear form of these equations is difficult, especially the analytic ones. Many researchers have been working to obtain analytic or approximate solutions of IEs and IDEs by applying different methods. Among these methods are Adomian decomposition (Singh and Wazwaz, 2016), variational iteration method (Yousefi, Lotfi and Dehghan, 2009; Wazwaz, 2010; Wang, He, 2007; Batiha, Noorani and Hashim, 2008), modified Laplace transform method (Majid, 2013), Sinc approximation (Maleknejad, Khalilsaraye and Alizadeh, 2013), modified homotopy perturbation method (Elbeleze, Klicman and Taib, 2016; Nadjafi, Ghorbani, 2009; Dolapci, Senol and Pakdemirli, 2009), Legendre spectral collocation method (Muhammad, Eshuvatov, Nurmuhammad, Mori and Sugihara, 2005; Sahu, Saha, 2015), weighted mean value theorem (Alturk, 2016) and many literature therein. (Liao, 2003) introduced another method called homotopy analysis method (HAM) to handle linear and nonlinear equations. HAM method gives solution in a series form. If the series solution obtained from HAM has a close form, the method provides exact solution. Otherwise, the solution is approximated to some degree of accuracy. Since the introduction of HAM several authors applied the method to solve linear and nonlinear IEs and IDEs. HAM gives freedom to control the convergence of the solution series by properly choosing the auxiliary things (initial guess $y_0(x)$, auxiliary parameter h, auxiliary linear operator L and auxiliary function H). Homotopy analysis method (HAM) for obtaining the numerical solutions of higher-order fractional integro-differential equations with boundary conditions was presented by (Zhang, Tang and He, 2011). The series solution was developed and the recurrence relations were given explicitly. The initial approximation can be freely chosen with possible unknown constants which can be determined by imposing the boundary conditions. The comparison of results obtained by the HAM with the exact solutions was made; the results reveal that the HAM is very effective and simple. The HAM contains the auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region of series solution. A hybrid method of

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Direct-homotopy analysis for solving fredholm integro-differential equations (DHAM) was proposed by (Bichi, Lawal, Lawan and Bello, 2018), the problem of integro-differential equation with separable kernel was considered. Convergence analysis to the exact solution of the proposed method was established. Examples were solved and comparisons were made with existing methods that shows the efficiency of the proposed method.
All these mentioned methods did not give way to control the convergence of the solution series. In this work, we consider homotopy analysis method which gives freedom to choose auxiliary linear operator, auxiliary parameter, auxiliary function and the initial guess. This freedom of choice provides a way to control the convergence of the solution series.

2. Description of Homotopy Analysis Method

Consider

$$N[y(t)] = 0, \tag{1}$$

where N is a nonlinear operator and $y(t)$ is the unknown function of independent variable t. For simplicity we ignore all initial or boundary conditions. (Liao, 2003) presented the so-called zero-order deformation equation as follows:

$$(1-q)L[\phi(t,q)-y_0(t)] = qhH(t)N[\phi(t,q)], \tag{2}$$

where $q \in [0, 1]$ is the embedding parameter, $h \neq 0$ an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L is a linear operator and $y_0(t)$ is the initial guess of $y(t)$. When $q = 0$ and $q = 1$ it holds that

$$\phi(t,0) = y_0(t) \text{ and } \phi(t,1) = y(t) \tag{3}$$

respectively. Thus, from (3) as q increases from 0 to 1 $\phi(t,q)$ varies from initial guess $y_0(t)$ to the solution $y(t)$. According to taylor's theorem $\phi(t,q)$ can be expanded in power of q as follows:

$$\phi(t,q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t)q^m, \tag{4}$$

where

$$y_m = \frac{1}{m!}\frac{\partial^m \phi(t,q)}{\partial q^m}\bigg|_{q=0}. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter and the auxiliary function are properly chosen, the series (4) converges at $q = 1$. So, we obtain

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{6}$$

Differentiating (2) m times with respect to the embedding parameter q and setting $q = 0$, then dividing through by $m!$, we obtain the mth order deformation equation as:

$$L[y_m(t) - \chi_m y_{m-1}(t)] = hH(t)R_m(y_{m-1}), \tag{7}$$

where

$$y_{m-1} = \{y_0(t), y_1(t), \dots, y_{m-1}(t)\}, \tag{8}$$

$$R_m(y_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(t,q)]}{\partial p^m - 1}\bigg|_{q=0}, \tag{9}$$

$$\chi_m = \begin{cases} 0, & \text{for } m \leq 1 \\ 1, & \text{for } m > 1. \end{cases} \tag{10}$$

If we cannot determine the sum of the series (6), then we accept the approximation of $y(t)$ as

$$y(t) \approx \sum_{m=0}^{n} y_m(t). \tag{11}$$

2.1. Mixed integro-differential equation (MIDHAM)

Consider the mixed integro-differential equation

$$u^n(x) = f(x) + \lambda \int_a^x \int_a^b K(x, t) u^{(l)}(t) dt dx, \tag{12}$$

$$u^{(r)}(0) = b_r, \quad 0 \le l \le r \le n-1, \tag{13}$$

where $K(x, t) = g(x)h(t)$. Integrating (12) n times from 0 to x gives

$$u(x) = F(x) + \lambda L^{-1}\left(\int_a^x g(x) \int_a^b h(t) u^{(l)}(t) dt dx\right) + \alpha, \tag{14}$$

where

$$L^{-1} = \int_0^x \int_0^x \cdots \int_0^x (\cdot) dx dx \cdots dx,$$

$$F(x) = L^{-1} f(x) \text{ and } \alpha = \sum_{r=1}^{n-1} b_{n-r} \frac{x^{n-r}}{(n-r)!} + b_0$$

we define the nonlinear operator as:

$$N[u(x)] = u(x) - F(x) + \alpha - \lambda L^{-1}\left(\int_a^x g(x) \int_a^b h(t) u^{(l)}(t) dt dx\right). \tag{15}$$

The corresponding *m*th-order deformation equation is

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x) R_m(u_{m-1}(x)), \tag{16}$$

where

$$\begin{aligned} R_m(u_{m-1}(x)) = {} & u_{m-1}(x) - (F(x) + \alpha)(1 - \chi_m) \\ & - \lambda L^{-1}\left(\int_a^x g(x)\left[\int_a^b h(t) u_{m-1}^{(l)}(t) dt\right] dx\right) \end{aligned} \tag{17}$$

Choosing the auxiliary linear parameter $L[u] = u$, we obtain from (16) and (17)

$$\begin{aligned} u_m(x) = {} & \chi_m u_{m-1}(x) + hH(x)\Bigg[u_{m-1}(x) - (F(x) - \alpha)(1 - \chi_m) \\ & - \lambda L^{-1}\left(\int_a^x g(x)\left[\int_a^b h(t) u_{m-1}^{(l)}(t) dt\right] dx\right)\Bigg] \end{aligned} \tag{18}$$

for $m = 1$

$$u_1(x) = hH(x)\left[u_0(x) - F(x) - \alpha \; - \lambda L^{-1}\left(\int_a^x g(x)\left[\int_a^b h(t) u_0^{(l)}(t) dt\right] dx\right)\right] \tag{19}$$

for $m > 1$ i.e., $m = 2, 3, 4, \ldots$

$$u_m(x) = u_{m-1}(x) + hH(x)\left[u_{m-1}(x) \; - \lambda L^{-1}\left(\int_a^x g(x)\left[\int_a^b h(t) u_{m-1}^{(l)}(t) dt\right] dx\right)\right] \tag{20}$$

with proper choice of initial guess $u_0(x)$, auxiliary function *H* and auxiliary parameter *h*, the series

$$u_0(x) + \sum_{m=1}^{\infty} u_m(x) \tag{21}$$

converges to the exact solution. And therefore the solution is

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \tag{22}$$

2.2. Convergence Analysis for DHAMVFIDE

In this section, we present the convergence of our proposed method. We prove as thus:

Theorem: *Suppose that* (22) *is convergent where* $u_0(x)$ *is the initial guess,* $u_1(x)$ *and* $u_m(x)$ *are obtained from* (19) *and* (20), *then it is the exact solution of* (12).

Proof. Let

$$u(x) = \sum_{m=0}^{\infty} u_m(x), \text{ be convergent,} \tag{23}$$

then it holds that

$$\lim_{m\to\infty} u_m(x) = 0 \tag{24}$$

we can see that

$$\sum_{m=1}^{n}[u_m(x) - \chi_m u_{m-1}(x)] \tag{25}$$

$$= u_1(x) + (u_2(x) - u_1(x)) + (u_3(x) - u_2(x)) + (u_n(x) - u_{n-1}(x)) \tag{26}$$

$$= u_n(x) \tag{27}$$

from (24) as $n \to \infty$

$$\sum_{m=1}^{\infty}[u_m(x) - \chi_m u_{m-1}(x)] \tag{28}$$

$$= \lim_{n\to\infty} \sum_{m=1}^{\infty}[u_m(x) - \chi_m u_{m-1}(x)] \tag{29}$$

$$= \lim_{n\to\infty} u_n(x) \tag{30}$$

$$= 0. \tag{31}$$

By definition of linear operator L

$$\sum_{m=1}^{\infty} L[u_m(x) - \chi_m u_{m-1}(x)] \tag{32}$$

$$= L\sum_{m=1}^{\infty}[u_m(x) - \chi_m u_{m-1}(x)] \tag{33}$$

$$= 0. \tag{34}$$

But

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)R_{m-1}(u_{m-1}(x)). \tag{35}$$

Thus we have from (32) and (35)

$$\sum_{m=1}^{\infty} L[u_m(x)-\chi_m u_{m-1}(x)] \tag{36}$$

$$=\sum_{m=1}^{\infty} hH(x)R_{m-1}(u_{m-1}(x)) \tag{37}$$

$$=hH(x)\sum_{m=1}^{\infty} R_{m-1}(u_{m-1}(x)) \tag{38}$$

$$=0. \tag{39}$$

Since $H \neq 0,\ h \neq 0,$ we have

$$\sum_{m=1}^{\infty} R_{m-1}(u_{m-1}(x))=0 \tag{40}$$

from (28) and (40)

$$0=\sum_{m=1}^{\infty} R_{m-1}(u_{m-1}(x)) \tag{41}$$

$$=\sum_{m=1}^{\infty}\left[u_{m-1}(x)-(F(x)+\alpha)(1-\chi_m)\ \ -\lambda L^{-1}\left(\int_a^x g(x)\left[\int_a^b h(t)u_{m-1}^{(l)}(t)dt\right]dx\right)\right] \tag{42}$$

$$=\sum_{m=1}^{\infty} u_{m-1}(x)-(F(x)+\alpha)-\lambda L^{-1}\left(\int_a^x g(x)\left[\int_a^b \sum_{m=1}^{\infty} h(t)u_{m-1}^{(l)}(t)dt\right]dx\right) \tag{43}$$

$$=\sum_{m=0}^{\infty} u_m(x)-(F(x)+\alpha)-\lambda L^{-1}\left(\int_a^x g(x)\left[\int_a^b h(t)\sum_{m=0}^{\infty} u_m^{(l)}(t)dt\right]dx\right) \tag{44}$$

thus

$$u(x)=F(x)+\alpha+\lambda L^{-1}\left(\int_a^x g(x)\int_a^b h(t)u_m^{(l)}(t)dt\right)dx$$

$$\simeq u(x)=F(x)+\alpha+\lambda L^{-1}\int_a^x\int_a^b K(x,\,t)u_m^{(l)}(t)dtdx \tag{45}$$

differentiating equation (45) *n* times, $u(x)$ must be the exact solution of equation (12) and

$$u^n(x)=\ f(x)+\lambda\int_a^x\int_a^b K(x,\,t)u^{(l)}(t)dtdx. \tag{46}$$

$\square$

3. Numerical Results

In this chapter, we present some numerical examples that use DHAMVFIDE to solve Volterra-Fredholm integro-differential equations. For unbiasedness, we make the choices of the auxiliary parameter $h=1$ and auxiliary function $H(x)=-1.$ The results obtained from DHAMVFIDE is compared with exact solution and absolute error.

Example 3.1. Consider the Volterra-Fredholm integro-differential equation

$$u'(x) = 6 + 29x - \frac{7}{2}x^2 + \int_0^x\int_0^1 (r-t)u(t)\,dt\,dr, \quad u(0) = 0 \tag{47}$$

the exact solution of the above equation is $u(x) = 6x + 12x^2.$ From equation (19) and equation (20)

$$F(x) = L^{-1}[f(x)] = \int_0^x 6 + 29x - \frac{7}{2}x^2 dx$$

$$= 6x + \frac{29}{2}x^2 - \frac{7}{6}x^3,$$

$$u(0) = b_0 = 0 \rightarrow \alpha = 0, \tag{48}$$

and

$$u_1(x) = -\left(u_0(x) - F(x) - \alpha - \int_0^x\left(\int_0^x\int_0^1 (r-t)u_0'(t)\,dt\,dr\right)dx\right), \tag{49}$$

$$u_m(x) = \int_0^x\left(\int_0^x\int_0^1 (r-t)u_{m-1}'(t)\,dt\,dr\right)dx, \quad m > 1. \tag{50}$$

Substituting for the value the initial guess $u_0(x)$, we used Maple to compute the absolute error at different values of *x* as shown in the table below:

Table 1. The exact solution, approximate solution obtained by DHAMVFIDE and absolute error obtained for Example 3.1

m	*x*	Exact solution	DHAMVFIDE 3.1	Absolute error
10	0.0	0.00	0.00000	0.0
	0.2	1.68	1.69100	1.10E-2
	0.4	4.32	4.29378	2.62E-2
	0.6	7.92	7.78577	1.34E-1
	0.8	12.48	12.24329	2.37E-1
20	0.0	0.00	0.00000	0.0
	0.2	1.68	1.69100	1.10E-2
	0.4	4.32	4.29378	2.62E-2
	0.6	7.92	7.78577	0.1.34E-1
	0.8	12.48	12.24329	2.37E-1

In Table 1 the exact solution of Example 3.1, the approximate solution obtained with our method of DHAMVFIDE and the absolute error are shown. The absolute error from the table indicates that DHAMVFIDE gives good approximation of Volterra-Fredholm integro-differential equation. The solution obtained with our method is approaching the exact solution as the error decreases when *m* gets large.

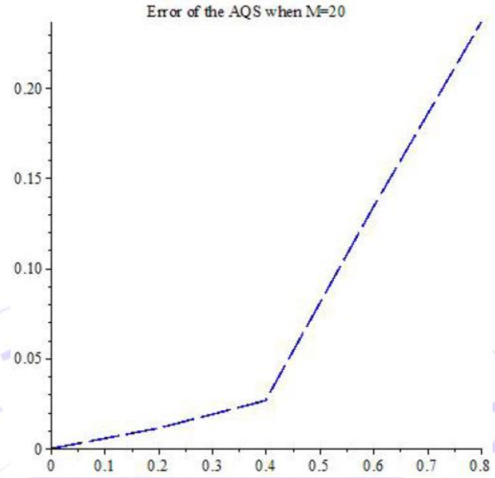

Figure 1. Error graph for Example 3.1 when $m = 20$.

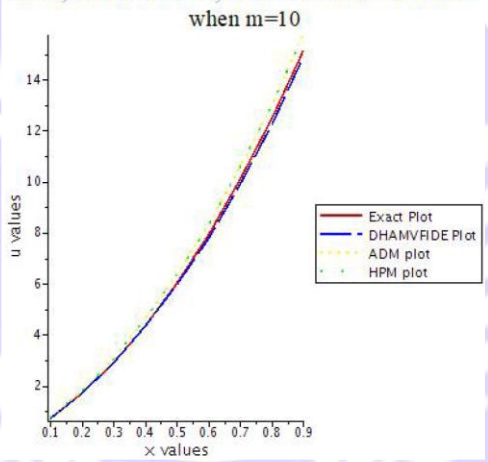

Figure 2. Comparison of absolute errors obtained by ADM, HPM and DHAMVFIDE for Example 3.1.

Example 3.2. Consider the Volterra-Fredholm integro-differential equation

$$u''(x) = -\frac{10}{3} + 2x^3 + \int_0^x \int_{-1}^1 (rt^2 - r^2 t) u(t) dt dr, \tag{51}$$

$$u(0) = 1, \quad u'(0) = 9 \tag{52}$$

the exact solution of the above equation is $u(x) = 1 + 9x - \frac{5}{3}x^2$. From equation (19) and equation (20)

$$F(x) = L^{-1}[f(x)] = \int_0^x \int_0^x -\frac{10}{3} + 2x^3 dx dx$$

$$= -\frac{5}{3}x^2 + \frac{1}{10}x^5,$$

$$b_0 = 1, \quad b_1 = 9 \rightarrow \alpha = 9x + 1, \tag{53}$$

and

$$u_1(x) = -\left(u_0(x) - F(x) - \alpha - \int_0^x \int_0^x \left(\int_0^x \int_{-1}^1 (rt^2 - r^2 t) u_0''(t) dt \right) dr \right) dx dx$$

(54)

$$u_m(x)=\int_0^x\int_0^x\Bigg(\int_0^x\int_{-1}^1(rt^2-r^2t)u''_{m-1}(t)dt\Bigg)dr\Bigg)dxdx\Bigg),\qquad m>1. \quad (55)$$

Substituting for the value the initial guess $u_0(x)$, we used Maple to compute the absolute error at different values of *x* as shown in the table below:

In Table 2 the exact solution of Example 3.2, the approximate solution obtained with our method of DHAMVFIDE and the absolute error are shown. The absolute error from the table indicates that DHAMVFIDE gives good approximation of Volterra-Fredholm integro-differential equation. The solution obtained with our method is approaching the exact solution as the error decreases when *m* gets large.

Table 2. The exact solution, approximate solution obtained by DHAMVFIDE and absolute error obtained for Example 3.2

m	*x*	Exact solution	DHAMVFIDE 3.2	Absolute error
10	0.0	1.00000	1.00000	0.00
	0.2	2.73333	2.73322	1.17E-4
	0.4	4.33333	4.33196	1.37E-3
	0.6	5.80000	5.79564	4.36E-3
	0.8	7.13333	7.12832	5.01E-3
20	0.0	1.00000	1.00000	0.00
	0.2	2.73333	2.73322	1.17E-4
	0.4	4.33333	4.33196	1.37E-3
	0.6	5.80000	5.79564	4.36E-3
	0.8	7.13333	7.12832	5.01E-3

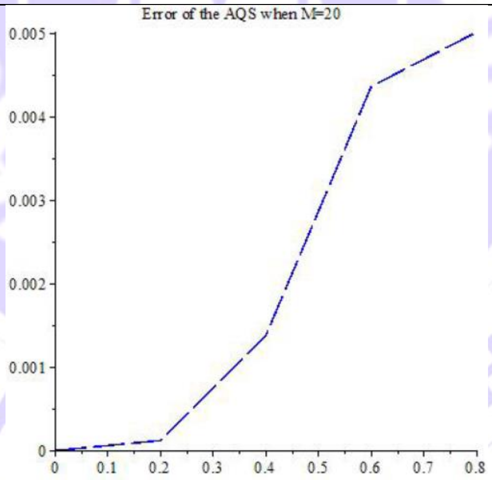

Figure 3. Error graph for Example 3.2 when $m=20$.

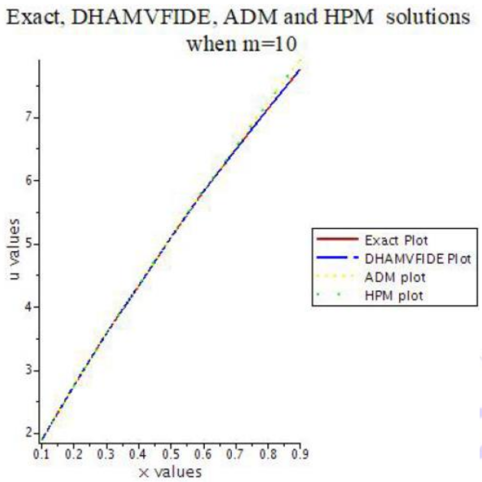

Figure 4. Comparison of absolute errors obtained by ADM, HPM and DHAMVFIDE for Example 3.2.

Example 3.3. Consider the Volterra-Fredholm integro-differential equation

$$u''(x) = -\frac{20}{3} + \frac{2}{3}x^3 + \int_0^x\int_{-1}^1 (rt^2 - r^2t)u(t)dtdr, \qquad (56)$$

$$u(0) = 1, \quad u'(0) = 9 \qquad (57)$$

the exact solution of the above equation is $u(x) = 2 + 2x - \frac{10}{3}x^2$. From equation (19) and equation (20)

$$F(x) = L^{-1}[f(x)] = \int_0^x\int_0^x -\frac{20}{3} + \frac{2}{3}x^3 dxdx$$

$$= -\frac{10}{3}x^2 + \frac{1}{30}x^5,$$

$$b_0 = 2, \quad b_1 = 3 \rightarrow \alpha = 3x + 2, \qquad (58)$$

and

$$u_1(x) = -\left(u_0(x) - F(x) - \alpha - \int_0^x\int_0^x\left(\int_0^x\int_{-1}^1 (rt^2 - r^2t)u_0''(t)dt\right)dr\right)dxdx \qquad (59)$$

$$u_m(x) = \int_0^x\int_0^x\left(\int_0^x\int_{-1}^1 (rt^2 - r^2t)u_{m-1}''(t)dt\right)dr\Big)dxdx\Big), \quad m > 1. \qquad (60)$$

Substituting for the value the initial guess $u_0(x)$, we used Maple to compute the absolute error at different values of *x* as shown in the table below: In Table 3 the exact solution of Example 3.3, the approximate solution obtained with our method of DHAMVFIDE and the absolute error are shown. The absolute error from the table indicates that DHAMVFIDE gives good approximation of Volterra-Fredholm integro-differential equation. The solution obtained with our method is approaching the exact solution as the error decreases when *m* gets large.

Table 3. The exact solution, approximate solution obtained by DHAMVFIDE and absolute error obtained for Example 3.3

m	*x*	Exact solution	DHAMVFIDE 3.3	Absolute error
	0.0	2.00000	2.00000	0.0
10	0.2	2.26667	2.46638	2.00E-1
	0.4	2.26667	2.66217	4.00E-1

	0.6	2.00000	2.57752	5.78E-1
	0.8	1.46667	2.19568	7.29E-1
20	0.0	2.00000	2.00000	0.0
	0.2	2.26667	2.46638	2.00E-1
	0.4	2.26667	2.66217	4.00E-1
	0.6	2.00000	2.57752	5.78E-1
	0.8	1.46667	2.19568	7.29E-1

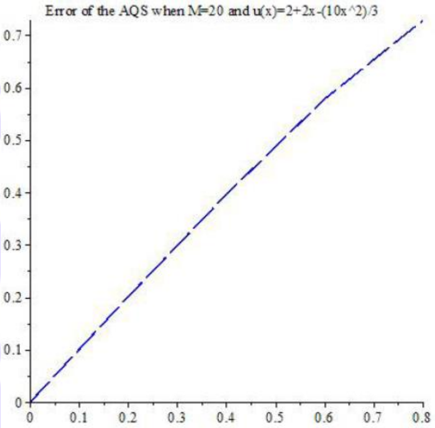

Figure 5. Error graph for Example 3.3 when $m = 10$.

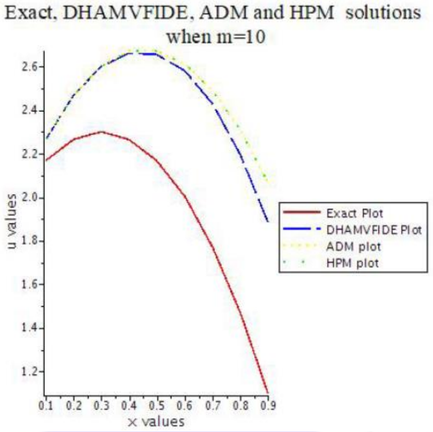

Figure 6. Comparison of absolute errors obtained by ADM, HPM and DHAMVFIDE for Example 3.3.

4. Conclusion

In this paper, a new method of DHAMVFIDE for solving Volterra-Fredholm integro-differential equations was developed. The method was tested with some examples and compared with some existing methods in the literature. The results obtained show that the method of DHAMVFIDE was effective and efficient for solving Volterra-Fredholm integro-differential equations. This research contributes to the field of Integro-differential equations by demonstrating their effectiveness and efficiency in solving problems of Mixed form. The development of new method and its application makes this study valuable addition to the existing literature.

However, the study had limitations, including the assumption of initial conditions and the use of assumptions in the numerical implementation. Future research should aim to relax these assumptions and explore the methods applicability to a wider range of problems. Potential areas of future research include extending to a more complex problem like nonlinear Volterra-Fredholm integro-differential equations and developing more efficient numerical algorithms for large-scale problems. By continuing to advance the field of integro-differential equations, researchers can develop more accurate and efficient methods for solving complex problems in science and engineering.

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