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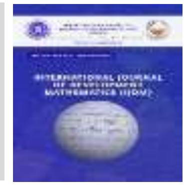

# On Some Classes of Regular Semigroups

**Abudulkarim Ibrahim[a], Udeogu Chidozie[b] and Babagana I. Bukar[c]**

[a]*Department of Basic Science and Tech., Mohamet Lawan College of Agriculture, Maiduguri, Borno State, Nigeria*
[b]*Department of Mathematics, Michael Okpara University of Agriculture Umudike, Abia State, Nigeria*
[c]*Department of Mathematics and Statistics, Yobe State University, Damaturu, Yobe State, Nigeria.*

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**ABSTRACT**

In this paper we study and analyze the relationships between some special classes of regular semigroups, namely: Generalized inverse semigroups, Orthodox semigroups and locally inverse semigroups based on their idempotent properties. We then use an example to prove that every locally inverse semigroup is not a generalized inverse semigroup, thus validating the statement; "A generalized inverse semigroup has the property of being locally inverse. However, not every locally inverse semigroup is a generalized inverse semigroup."

## 1. Introduction

Garba (2019) stated "Semigroup theory is a relatively young part of mathematics, as a special direction of algebra with its own objects, formulations of problems and methods of investigations. The main reason that semigroups turn up in Mathematics is that one is very often interested in self-maps of a set of one kind or another, and whenever $f, g, h$ are such maps, it is automatically the case that $fo(goh) = (fog)oh$. If the maps are bijections then the appropriate abstract idea is that of a group; if not then inevitably we must consider a semigroup."

By a semigroup we simply mean any algebraic structure that satisfies the closure and associative property of a binary operation. Obvious examples of semigroups are the natural numbers, $\mathbb{N}$ under either addition or multiplication. If a semigroup$S$ satisfies the property that for all $x, y \in S$ we have $xy = yx$, then we say that $S$ is a commutative semigroup.

The main object of interest in this research work is one of the special classes of semigroups called regular semigroups which were introduced by Green in 1951 in his influential paper "On the structure of semigroups." Regular semigroups can be considered as the core semigroups since groups are regular semigroups with a unique idempotent. The idempotents play a predominant role in the structure of regular semigroups (Julie *et al*., 2018).

Different classes of regular semigroups were defined based on some properties of idempotents: Inverse semigroups, orthodox semigroups, locally inverse semigroups, generalized inverse semigroups etc. A semigroup $S$ is an inverse semigroup if and only if it is regular and its idempotents commute. Inverse semigroups were introduced independently by Wagner in 1952 and by Preston in 1954. Locally inverse semigroups and orthodox semigroups are regular generalizations of inverse semigroups (Julie *et al*., 2018). Orthodox semigroups are due to Hall (1969) and Yamada (1970). Yamada (1967) gave a complete classification of generalized inverse semigroups in terms of inverse semigroups, left normal bands, and right normal bands. Indhira and Chandrasekaran (2011) established a new structure theorem for generalized inverse semigroups and specialized it to a generalized Brandt semigoups.

In this paper however, we study three classes of regular semigroups: generalized inverse semigroups, locally

* Corresponding author. Tel.:0706 537 8093
E-mail address: udeogu.chidozie@mouau.edu.ng (Udeogu, C.)
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inverse semigroups and orthodox semigroups and their relationships. We then present a concrete example showing why a locally inverse semigroup is not a generalized inverse semigroup despite every generalized inverse semigroup being locally inverse.

## 2. Preliminaries

**Definition 2.1 (Howie, 1995):** A semigroup $\mathcal{S}$ is called a monoid if it contains an identity, otherwise we can easily adjoin an extra identity 1 in order to make it monoid. Thus, we write $\mathcal{S}^1$ and $\mathcal{S}^0$ respectively to denote a semigroup $\mathcal{S}$ to which each 1 or 0 is adjoined.

That is :

$$\mathcal{S}^1 = \begin{cases} \mathcal{S} & if\ S\ has\ identity\ element \\ \mathcal{S} \cup \{1\} & otherwise \end{cases}$$

where $1x = x1 = x, \ \forall\, x \in \mathcal{S}$.

and

$$\mathcal{S}^0 = \begin{cases} \mathcal{S} & if\ S\ has\ identity\ element \\ \mathcal{S} \cup \{0\} & otherwise \end{cases}$$

where $0x = x0 = 0, \ \forall\, x \in \mathcal{S}$.

**Definition 2.2 (Howie, 1995):** An element $a$ of a semigroup $\mathcal{S}$ is called *regular* if there exist an element $x$ in $\mathcal{S}$ such that $axa = a$. A semigroup $\mathcal{S}$ is said to be a *regular semigroup* provided every of its elements are regular.

**Definition 2.3 (Howie, 1995):** An element '$\ell$' of a semigroup $\mathcal{S}$ is called an*idempotent* if $\ell^2 = \ell$. If every element of $\mathcal{S}$ is an idempotent then $\mathcal{S}$ is called an *idempotent semigroup* or a *band*. We denote the set of all idempotents in $\mathcal{S}$ by $\mathcal{E}(\mathcal{S})$.

**Example 2.1:** Let $\ell = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$ be element in $T_3$, the full transformation semigroup on three symbols, then we can see that $\ell$ is idempotent since,

$$\ell^2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} = \ell$$

**Definition 2.4 (Howie, 1995):** Let $\mathcal{S}$ be a regular semigroup. Then for element $a \in \mathcal{S}$, if there exists an element $\hat{a}$ such that $a\hat{a}a = a$ and $\hat{a}a\hat{a} = \hat{a}$, such an element $\hat{a}$ is called an *inverse* for $a$.

A semigroup $\mathcal{S}$ is an *inverse semigroup* if each element in $\mathcal{S}$ has a unique inverse in $\mathcal{S}$. In terms of idempotents, a semigroup $\mathcal{S}$ is an inverse semigropup if and only if it is regular and its idempotents commute.

**Definition 2.5 (Guo, 2008):** A regular semigroup $\mathcal{S}$ with set $\mathcal{E}$ of idempotents is called *locally inverse* if $\ell\mathcal{S}\ell$ is an inverse semigroup for each $\ell$ in $\mathcal{E}$

**Definition 2.6 (Howie, 1995):** A regular semigroup $\mathcal{S}$ is called an *orthodox* if the set of all idempotents $\mathcal{E}(\mathcal{S})$ forms a subsemigroup.

**Definition 2.7 (Howie, 1995):** A *generalized inverse semigroup* is a regular semigroup in which the set of idempotents satisfies a permutation identity $x_1, x_2 \dots x_n = x_{p_1}, x_{p_2} \dots x_{p_n}$, where ( $p_1, p_2 \dots p_n$) is a non-trivial permutation of $(1, 2, \dots, n)$.

**Definition 2.8 (Howie, 1995):** A semigroup $\mathcal{S}$ is said to be *normal* if its satisfies the identity $abca = acba$ for all $a, b, c \in \mathcal{S}$.

**Lemma 2.1 (Howie, 1995):** Let $\mathcal{S}$ be a semigroup. Then the following conditions are equivalent:

(i) $\mathcal{S}$ is an inverse semigroup
(ii) $\mathcal{S}$ is regular and its idempotents commutes
(iii) Every $\mathcal{L}$- class and every $\mathcal{R}$- class contains exactly one idempotent.
(iv) Every element of $\mathcal{S}$ has unique inverse.

**Theorem 2.1 (Howie, 1995):** Let $\mathcal{S}$ be a regular semigroup with set $\mathcal{E}$ of idempotents. Then the following statements are equivalent:

(i) $\mathcal{S}$ is orthodox
(ii) For any $a, b$ in $\mathcal{S}$, if $\hat{a}$ is an inverse of $a$ and $\hat{b}$ is an inverse $b$, then $\hat{b}\hat{a}$ is an inverse of $ab$;
(iii) If $\ell$ is idempotent then every inverse of $\ell$ is idempotent.

**Lemma 2.2 (Howie, 1995):** A band $\mathcal{E}$ is normal if and only if it satisfies the identity
$xyzx = xzyx \ or \ xyxzx = xzxyx$
for every $x, y, z \in \mathcal{E}$.

**Lemma 2.3 (Howie, 1995):** If $\ell, f, \ell f$, and $f\ell$ are all idempotent elements of a semigroup, then $\ell f$ and $f\ell$ are inverses of each other.

Here, we now describe the relationship between generalized inverse semigroups, locally inverse semigroups and orthodox semigroups by the following results due to Yamada [13]:

**Lemma 2.4 (Yamada, 1967):** Let $\mathcal{S}$ be a regular semigroup in which the set $\mathcal{E}$ of idempotents is a normal band. Then for any elements $a, b$ of $\mathcal{S}$ and for any elements $\ell, f$ of $\mathcal{E}$,
$$a\ell f b = af\ell b. \qquad \text{……….(*)}$$

**Lemma 2.5 (Yamada, 1967):** (1) If $\mathcal{S}$ is a regular semigroup such that for any elements $a, b$ of $\mathcal{S}$ the element $\hat{b}\hat{a}$ is an inverse of $ab$, for any inverses $\hat{a}$ of $a$ and $\hat{b}$ of $b$(the product inverse property), then the set of idempotents $\mathcal{E}$ of $\mathcal{S}$ is a band.
(2) If the set of idempotents of a regular semigroup $\mathcal{S}$ is a normal band, then the product inverse property follows, i.e. for any elements $a, b$ of $\mathcal{S}$ and for any inverses $\hat{a}$ of $a$ and $\hat{b}$of $b$, the element $\hat{b}\hat{a}$ is an inverse of $ab$.

**Lemma 2.6 (Yamada, 1967):** If the set $\mathcal{E}$ of idempotents of a regular semigroup $\mathcal{S}$ satisfies a permutation identity, then $\mathcal{E}$ is a normal band.

**Lemma 2.7 (Yamada, 1967):** Let $\mathcal{S}$ be a regular semigroup in which the set $\mathcal{E}$ of idempotents is a band.
(1) If $\mathcal{E}$ is a normal band, then the intersection $a\mathcal{S} \cap \mathcal{S}b (= a\mathcal{S}b)$ of a principal right ideal $a\mathcal{S}$ and a principal left ideal $\mathcal{S}b$ is a subsemigroup in which any two of the idempotents commute. In particular, $\ell\mathcal{S}\ell$ is an inverse semigroup for any idempotent $\ell$ of $\mathcal{S}$.
(2) If every $\ell f \ell$ has precisely one inverse in $\ell\mathcal{S}\ell$, then $\mathcal{E}$ is a normal band, where $\ell$ and $f$ are elements of $\mathcal{E}$.

**Theorem 2.8 (Yamada, 1967):** The following five conditions on a regular semigroup $\mathcal{S}$ are equivalent:
(1) $\mathcal{S}$ is a generalized inverse semigroup;
(2) The set of idempotents of $\mathcal{S}$ is a normal band;
(3) The set of idempotents of $\mathcal{S}$ is a band, and the intersection $a\mathcal{S} \cap \mathcal{S}b (= a\mathcal{S}b)$ of a principal right ideal $a\mathcal{S}$

and a principal left ideal $\mathcal{S}\mathscr{b}$ is a subsemigroup in which any two of the idempotents commute;
(4) The set of idempotents of $\mathcal{S}$ is a band, and $\ell\mathcal{S}\ell$ is an inverse subsemigroup for any idempotent $\ell$ of $\mathcal{S}$;
(5) $\mathcal{S}$ satisfies the product inverse property. Further, every $\ell\mathscr{f}\ell$, where $\ell$ and $\mathscr{f}$ are idempotents of $\mathcal{S}$, has precisely one inverse in $\ell\mathcal{S}\ell$.

## 3. Relationships between the Three Classes of Regular Semigroups

### 3.1 Discussions

Lemma 2.4 expresses the fact that when the set of idempotents of a regular semigroup is a normal band, the relation (*) must be satisfied.

In Lemma 2.5(1) we see that if in a regular semigroup, the product inverse property is satisfied, then the set of idempotents of the semigroup is a band (subsemigroup). In other words, the semigroup is an orthodox semigroup.

Lemma 2.5(2) presents the fact that if the set of idempotents is now a normal band, then it automatically satisfies the inverse product property.

Lemma 2.6 now asserts that whenever the set of idempotents of a regular semigroup satisfies the permutation identity, it is automatically a normal band. In other words, the regular semigroup is now a generalized inverse semigroup.

Hence from Lemmas 2.5(1 and 2) to Lemma 2.6, we can conclude that a generalized inverse semigroup $\mathcal{S}$ is a normal band if it satisfies the permutation identity, and by being a normal band, it satisfies the inverse property. Since it satisfies the inverse property, it is a band, which in turn implies that the semigroup is an orthodox semigroup.

Next, by Lemma 2.7(1) we have that if the orthodox semigroup $\mathcal{S}$ is a normal band, then $\ell\mathcal{S}\ell$ is an inverse semigroup for any idempotent $\ell$, thus, the semigroup $\mathcal{S}$ is now a locally inverse semigroup. And Lemma 2.7(2) makes it clear that whenever $\ell\mathcal{S}\ell$ is an inverse semigroup for any idempotent $\ell$, the set of idempotents is a normal band.

These relationships between the three classes of regular semigroups we just analyzed can be said to be revolving around the set of idempotents being a normal band.

Hence, Indhira and Chandrasekaran (2011) summarized the whole relationships in the following way:

**Lemma 3.1 (Indhira and Chandrasekaran, 2011):** Let $\mathcal{S}$ be an orthodox and locally inverse semigroup. Then the set of idempotents of $\mathcal{S}$ is a normal band.
**Lemma 3.2 (Indhira and Chandrasekaran, 2011):** If the set of idempotents $\mathcal{E}$ of a regular semigroup $\mathcal{S}$ satisfies a permutation identity then $\mathcal{E}$ is a normal band.

The following theorem gives a characterization of generalized inverse semigroup

**Theorem 3.3 (Indhira and Chandrasekaran, 2011):** Let $\mathcal{S}$ be a regular semigroup. Then the following are equivalent.

a) $\mathcal{S}$ is a generalized inverse semigroup.
b) The set of idempotents of $\mathcal{S}$ is a normal band.
c) $\mathcal{S}$ is orthodox and locally inverse semigroup.

We now present a concrete example showing that a locally inverse semigroup is not a generalized inverse semigroup. This is to support the comment made by Howie (1965) page 222.

**Example 3.1:** To show that a locally inverse semigroup is not a generalized inverse semigroup, it will suffice to present a regular semigroup that is locally inverse and has idempotent elements whose product is not idempotent.
Thus, let $\mathcal{S} = \{\mathscr{p}, 0, \mathscr{q}, \mathscr{w}, \mathscr{pq}\}$ be a semigroup with zero adjoined, whose table is given below

| | $p$ | 0 | $q$ | $w$ | $pq$ |
|---|---|---|---|---|---|
| $p$ | $p$ | 0 | $pq$ | $p$ | $pq$ |
| 0 | 0 | 0 | 0 | 0 | 0 |
| $q$ | 0 | 0 | $q$ | $w$ | 0 |
| $w$ | $w$ | 0 | $q$ | $w$ | $q$ |
| $pq$ | $p$ | 0 | 0 | $p$ | 0 |

It is clear that $S$ is regular, locally inverse but the product of its idempotents $p$ and $q$ is not idempotent. Thus, $S$ is not orthodox. Hence, $S$ is not a generalized inverse semigroup.

## 4. Conclusion

In this paper we have studied regular semigroups, specifically, three of its classes: generalized inverse semigroups, orthodox semigroups and locally inverse semigroups. We analyzed the relationships between them as shown in the results by Yamada (1967) and summarized it using the results by Indhira and Chandrasekaran (2011) (page 741). From the relationships we analyzed, it is clear that every generalized inverse semigroup is locally inverse but the converse is not true as stated in Howie (1965), page 222, but without a proof. So, in this work we provided a proof in terms of an example to support the statement.

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