International Journal of Development Mathematics Vol 1 Issue 1 Page No. 72 - 89

INTERNATIONAL JOURNAL OF DEVELOPMENT MATHEMATICS

ISSN: 3026-8656 (Print) | 3026-8699 (Online)

journal homepage: https://ijdm.org.ng/index.php/Journals

An Efficient Scheme for Direct Simulation of Third and Fourth Oscillatory Differential Equations

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ARTICLE INFO

Article history:

Received 21 December 2024

Received in revised form 14 February 2024

Accepted 27 February 2024

Keywords:

Convergence, Consistency, Direct simulation, Linear stability, Oscillatory differential equations

MSC 2020 Subject classification:

12H20, 34C60

ABSTRACT

In this research, we've explored the general block approach for solving higher-order oscillatory differential equations using the linear block approach (LBA). Additionally, we've analysed and satisfied the basic properties of the new method, such as order, error constant, zero-stability, consistency, convergence, linear stability, and region of absolute stability. To overcome the setbacks of the reduction method, we directly applied some distinct fourth-order oscillatory problems to the method. The results were compared with those in the literature, demonstrating that the proposed method alleviates the burden of solving fourth-order oscillatory differential equations. Consequently, the new method has exhibited better accuracy and faster convergence graphically. One advantage of the new method is its minimal computational burden and self-starting capability.

1. Introduction

Many physical problems remain unexplored and not yet fully addressed by researchers. While some problems in the fields of science, social science, and technology have been approached, many others remain uncharted territory. Oscillatory phenomena often play a key role in these areas, and one of the primary tools for modeling such oscillations is through the use of differential equations (Sabo *et al*. (2022)).

The direct solution of fourth order oscillatory differential equation of the form

$$y^{iv}(t) = f(t, y, y', y'', y'''),\ y^{(z)}(t_0) = y_z,\ z = 0(1)3,\ t \in [t_0, b] \qquad (1)$$

is considered in this manuscript. Several phenomena in the physical or real life, such as the oscillatory differential equation in ship dynamics, inhomogeneous nonlinear systems, and neutral networks, can be expressed as (1) (Gear (1966), Gear (1971) and Lambert (1973)). However, the results of solving (1) may not be achieved directly (Gear (1978) and Suleiman (1979)). Hence, it is better to develop numerical methods in block form to address the issue (Suleiman (1989), Hall and Suleiman (1981) and Kayode (2008)).

Conventional methods for solving higher-order ordinary differential equations involve reducing them to a system of first-order ordinary differential equations. Subsequently, appropriate numerical methods for first-order equations are applied to solve the system (Lambert (1973)). This approach calculates the numerical solution one point at a time. Nevertheless, its significant drawbacks include computational burdens and complexities, which compromise accuracy by introducing errors. Additionally, challenges arise in programming the method and in its time-consuming nature. To address these challenges and enhance numerical methods, a novel one-step block method will be developed, featuring eight partitions for the direct solution of higher-order initial value problems ((Suleiman (1989)). This method will specifically target problems arising in physics, biology, chemistry, and economics (Hall and Suleiman (1981)). The new method will be employed to address second-order problems such as simple harmonic motion and dynamic mass in motion, third-order challenges including linear and nonlinear oscillatory differential equations, and fourth-order scenarios such as thin film dynamics and highly stiff linear and nonlinear oscillatory differential equations.

Researchers have employed oscillatory differential equations to tackle complex systems involving multiple variables ((Abolarin *et al*. (2022)). Some authors who worked on hybrid block methods, among others, are (Donald, *et al*. (2021), Ayinde *et al*. (2023), Kuboye, *et al*. (2022). This field of study is of great significance to numerical analysts as it enables the simulation of various phenomena in the realms of science, engineering, and social sciences (Areo and Rufai (2016) and Olabode and Momoh (2016)). For instance, it provides solutions for problems related to

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https://doi.org/10.62054/ijdm/0101.07

transportation, mass-spring systems, simple harmonic motion, and dynamic systems of objects, among others (Ayinde *et al.* (2023), Kuboye, *et al*. (2022)).

The block method is one of the most efficient direct methods available. The block method was proposed in 1953 by (Adoghe *et al*. (2016)) to find the solution of the numerical method at more than one point simultaneously. In 1967, the block method was employed to provide the essential starting value needed for the predictor-corrector method (Modebeia *et al*. (2021). In 1964, the hybrid method was initiated by (Duromola (2022)) to evaluate the functions at off-step (non-step) points. The hybrid block methods were introduced according to (Sabo *et al*. (2022)) to change the step size and off-step points and to evade the zero-stability barrier condition (Abolarin *et al*. (2022)). Some authors who worked on hybrid block methods, among others, are (Donald, *et al*. (2021), Ayinde et al. (2023), Kuboye, *et al*. (2022) and Adoghe *et al*. (2019)). The numerical results reveal that the accuracy of the methods is better than the previous hybrid methods. The methods of (Lambert (1973), Sabo *et al*. (2021 Raymond *et al*. (2023), Wend (1969), Areo and Rufai (2016) and Olabode and Momoh (2016)) were verified on some numerical examples, and the accuracy of the methods can still be improved in terms of error.

In the literature (Agarwal *et al*. (2013), Awoyemi (2003) and Blanka (2019)), it has been observed that equation (1.1) can be solved by reducing the system to its first-order system of the form

$$y^{i}(t) = f(t, y) \tag{2}$$

However, the researchers suffer from some setbacks, such as the evaluation of too many functions, and as a result, more work is needed with heavy computation (Kwari *et al*. (2023)). The direct methods of solving (1) opted to circumvent this drawback. The direct methods proposed by (Donald, *et al*. (2021), Ayinde *et al*. (2023) and Sabo *et al*. (2021)).

Following in the scholar's footsteps, the direct simulation of a higher-order oscillatory differential equation using a linear block approach will be proposed in this research.

2. Formulation of the Higher-Order Block Approach

2.1 The $k-step$ Generalized algorithm

The numerical approach adopted was the linear block approach for the direct solution of higher order oscillatory differential equation (1) where $Y_{n+k} = \left(y_{n+a}, y_{n+b}, \cdots, y_{n+k}\right)$ and $Y_{n+k}^{(j)} = \left(y_{n+a}^{(j)}, y_{n+b}^{(j)}, \cdots, y_{n+k}^{(j)}\right)$.

In order to obtain the unknown values that the generalized algorithm

$$y_{n+\zeta} = \sum_{j=0}^{3} \frac{(\zeta h)^{j}}{j!} y_{n}^{(j)} + \sum_{j=0}^{k} \left(\rho_{i\zeta} f_{n+j}\right), \quad \zeta = a, b, \cdots, k \tag{3}$$

and its higher derivatives

$$y_{n+\zeta}^{q} = \sum_{j=0}^{4-(q+1)} \frac{(\zeta h)^{j}}{j!} y_{n}^{(j+q)} + \sum_{j=0}^{k} \left(\vartheta_{\zeta\, jq} f_{n+j}\right), \quad q = 1_{(\zeta=a,b,\cdots,k)}, q = 2_{(\zeta=a,b,\cdots,k)}, q = 3_{(\zeta=a,b,\cdots,k)} \tag{4}$$

$\rho_{\zeta j} = \mathrm{X}^{-1}\mathrm{A}$ and $\vartheta_{\zeta\, jq} = \mathrm{X}^{-1}\mathrm{N}$ where

$$\mathrm{X} = \begin{pmatrix} 1 & 1 & 1 & \cdots & k \\ 0 & \frac{(ah)^{1}}{1!} & \frac{(bh)^{1}}{1!} & \cdots & \frac{(kh)^{1}}{1!} \\ 0 & \frac{(ah)^{2}}{2!} & \frac{(bh)^{2}}{2!} & \cdots & \frac{(kh)^{2}}{2!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{(ah)^{n}}{n!} & \frac{(bh)^{n}}{n!} & \cdots & \frac{(kh)^{n}}{n!} \end{pmatrix}, \mathrm{A} = \begin{pmatrix} \frac{(\tau h)^{4}}{4!} \\ \frac{(\tau h)^{5}}{5!} \\ \frac{(\tau h)^{6}}{6!} \\ \vdots \\ \frac{(\tau h)^{(4+n)}}{(4+n)!} \end{pmatrix}, \mathrm{N} = \begin{pmatrix} \frac{(\tau h)^{4-q}}{(4-q)!} \\ \frac{(\tau h)^{(5-q)+a}}{((5-q)+a)!} \\ \frac{(\tau h)^{(6-q)+b}}{((6-q)+b)!} \\ \vdots \\ \frac{(\tau h)^{(4-q)+n}}{((4-q)+n)!} \end{pmatrix}$$

The following Corollary was considered in order to derive the new schemes.

Corollary 1

The general linear multistep method adopts only one block form for every $k-step$ block method and the corollary seen below is generalized to develop the fourth order scheme from linear block algorithm.
This can be verified with the help of the equation (3) and (4) as a block using the partitions $\left(0,\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{3}{4},\frac{7}{8},1\right)$.

Proof

Simplifying (3) and (4) using the partitioned points, we have the system

$$X=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & \frac{h}{8} & \frac{h}{4} & \frac{3h}{8} & \frac{h}{2} & \frac{5h}{8} & \frac{3h}{4} & \frac{7h}{8} & h\\
0 & \frac{\left(\frac{h}{8}\right)^2}{2!} & \frac{\left(\frac{h}{4}\right)^2}{2!} & \frac{\left(\frac{3h}{8}\right)^2}{2!} & \frac{\left(\frac{h}{2}\right)^2}{2!} & \frac{\left(\frac{5h}{8}\right)^2}{2!} & \frac{\left(\frac{3h}{4}\right)^2}{2!} & \frac{\left(\frac{7h}{8}\right)^2}{2!} & \frac{(h)^2}{2!}\\
0 & \frac{\left(\frac{h}{8}\right)^3}{3!} & \frac{\left(\frac{h}{4}\right)^3}{3!} & \frac{\left(\frac{3h}{8}\right)^3}{3!} & \frac{\left(\frac{h}{2}\right)^3}{3!} & \frac{\left(\frac{5h}{8}\right)^3}{3!} & \frac{\left(\frac{3h}{4}\right)^3}{3!} & \frac{\left(\frac{7h}{8}\right)^3}{3!} & \frac{(h)^3}{3!}\\
0 & \frac{\left(\frac{h}{8}\right)^4}{4!} & \frac{\left(\frac{h}{4}\right)^4}{4!} & \frac{\left(\frac{3h}{8}\right)^4}{4!} & \frac{\left(\frac{h}{2}\right)^4}{4!} & \frac{\left(\frac{5h}{8}\right)^4}{4!} & \frac{\left(\frac{3h}{4}\right)^4}{4!} & \frac{\left(\frac{7h}{8}\right)^4}{4!} & \frac{(h)^4}{4!}\\
0 & \frac{\left(\frac{h}{8}\right)^5}{5!} & \frac{\left(\frac{h}{4}\right)^5}{5!} & \frac{\left(\frac{3h}{8}\right)^5}{5!} & \frac{\left(\frac{h}{2}\right)^5}{5!} & \frac{\left(\frac{5h}{8}\right)^5}{5!} & \frac{\left(\frac{3h}{4}\right)^5}{5!} & \frac{\left(\frac{7h}{8}\right)^5}{5!} & \frac{(h)^5}{5!}\\
0 & \frac{\left(\frac{h}{8}\right)^6}{6!} & \frac{\left(\frac{h}{4}\right)^6}{6!} & \frac{\left(\frac{3h}{8}\right)^6}{6!} & \frac{\left(\frac{h}{2}\right)^6}{6!} & \frac{\left(\frac{5h}{8}\right)^6}{6!} & \frac{\left(\frac{3h}{4}\right)^6}{6!} & \frac{\left(\frac{7h}{8}\right)^6}{6!} & \frac{(h)^6}{6!}\\
0 & \frac{\left(\frac{h}{8}\right)^7}{7!} & \frac{\left(\frac{h}{4}\right)^7}{7!} & \frac{\left(\frac{3h}{8}\right)^7}{7!} & \frac{\left(\frac{h}{2}\right)^7}{7!} & \frac{\left(\frac{5h}{8}\right)^7}{7!} & \frac{\left(\frac{3h}{4}\right)^7}{7!} & \frac{\left(\frac{7h}{8}\right)^7}{7!} & \frac{(h)^7}{7!}\\
0 & \frac{\left(\frac{h}{8}\right)^8}{8!} & \frac{\left(\frac{h}{4}\right)^8}{8!} & \frac{\left(\frac{3h}{8}\right)^8}{8!} & \frac{\left(\frac{h}{2}\right)^8}{8!} & \frac{\left(\frac{5h}{8}\right)^8}{8!} & \frac{\left(\frac{3h}{4}\right)^8}{8!} & \frac{\left(\frac{7h}{8}\right)^8}{8!} & \frac{(h)^8}{8!}
\end{pmatrix},\ A=\begin{pmatrix}
\frac{(\zeta h)^4}{4!}\\ \frac{(\zeta h)^5}{5!}\\ \frac{(\zeta h)^6}{6!}\\ \frac{(\zeta h)^7}{7!}\\ \frac{(\zeta h)^8}{8!}\\ \frac{(\zeta h)^9}{9!}\\ \frac{(\zeta h)^{10}}{10!}\\ \frac{(\zeta h)^{11}}{11!}\\ \frac{(\zeta h)^{12}}{12!}
\end{pmatrix},\ N=\begin{pmatrix}
\frac{(\zeta h)^{4-\iota}}{(4-\iota)!}\\ \frac{(\zeta h)^{5-\iota}}{(5-\iota)!}\\ \frac{(\zeta h)^{6-\iota}}{(6-\iota)!}\\ \frac{(\zeta h)^{7-\iota}}{(7-\iota)!}\\ \frac{(\zeta h)^{8-\iota}}{(8-\iota)!}\\ \frac{(\zeta h)^{9-\iota}}{(9-\iota)!}\\ \frac{(\zeta h)^{10-\iota}}{(10-\iota)!}\\ \frac{(\zeta h)^{11-\iota}}{(11-\iota)!}\\ \frac{(\zeta h)^{12-\iota}}{(12-\iota)!}
\end{pmatrix}$$

Solving equations (3) and (4), we obtain the coefficients of the polynomial $y_{\tau\zeta},\ \zeta=0,\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{3}{4},\frac{7}{8},1$

Substituting $t=t_\tau+xh$, the polynomial takes the form

$$y(t_\tau+xh)=\alpha_{\frac{1}{8}}y_{\tau+\frac{1}{8}}+\alpha_{\frac{3}{8}}y_{\tau+\frac{3}{8}}+\alpha_{\frac{5}{8}}y_{\tau+\frac{5}{8}}+\alpha_{\frac{7}{8}}y_{\tau+\frac{7}{8}}$$
$$+h^4\left(\beta_0 f_\tau+\beta_{\frac{1}{8}}f_{\tau+\frac{1}{8}}+\beta_{\frac{1}{4}}f_{\tau+\frac{1}{4}}+\beta_{\frac{3}{8}}f_{\tau+\frac{3}{8}}+\beta_{\frac{1}{2}}f_{\tau+\frac{1}{2}}+\beta_{\frac{5}{8}}f_{\tau+\frac{5}{8}}+\beta_{\frac{3}{4}}f_{\tau+\frac{3}{4}}+\beta_{\frac{7}{8}}f_{\tau+\frac{7}{8}}+\beta_1 f_{\tau+1}\right) \tag{5}$$

The linear block approach (3) is expanded to yield

$$\left.\begin{array}{l}
y_{n+\frac{1}{8}} = y_n + \frac{1}{8}hy'_n + \frac{\left(\frac{1}{8}h\right)^2}{2!}y''_n + \frac{\left(\frac{1}{8}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\rho_{01}f_\tau + \rho_{02}f_{\tau+\frac{1}{8}} + \rho_{03}f_{\tau+\frac{1}{4}} + \rho_{04}f_{\tau+\frac{3}{8}} + \\ \rho_{05}f_{\tau+\frac{1}{2}} + \rho_{06}f_{\tau+\frac{5}{8}} + \rho_{07}f_{\tau+\frac{3}{4}} + \rho_{08}f_{\tau+\frac{7}{8}} + \rho_{09}f_{\tau+1}\end{pmatrix} \\
y_{n+\frac{1}{4}} = y_n + \frac{1}{4}hy'_n + \frac{\left(\frac{1}{4}h\right)^2}{2!}y''_n + \frac{\left(\frac{1}{4}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\rho_{01}f_\tau + \rho_{02}f_{\tau+\frac{1}{8}} + \rho_{03}f_{\tau+\frac{1}{4}} + \rho_{04}f_{\tau+\frac{3}{8}} + \\ \rho_{05}f_{\tau+\frac{1}{2}} + \rho_{06}f_{\tau+\frac{5}{8}} + \rho_{07}f_{\tau+\frac{3}{4}} + \rho_{08}f_{\tau+\frac{7}{8}} + \rho_{09}f_{\tau+1}\end{pmatrix} \\
y_{n+\frac{3}{8}} = y_n + \frac{3}{8}hy'_n + \frac{\left(\frac{3}{8}h\right)^2}{2!}y''_n + \frac{\left(\frac{3}{8}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\rho_{01}f_\tau + \rho_{02}f_{\tau+\frac{1}{8}} + \rho_{03}f_{\tau+\frac{1}{4}} + \rho_{04}f_{\tau+\frac{3}{8}} + \\ \rho_{05}f_{\tau+\frac{1}{2}} + \rho_{06}f_{\tau+\frac{5}{8}} + \rho_{07}f_{\tau+\frac{3}{4}} + \rho_{08}f_{\tau+\frac{7}{8}} + \rho_{09}f_{\tau+1}\end{pmatrix} \\
y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy'_n + \frac{\left(\frac{1}{2}h\right)^2}{2!}y''_n + \frac{\left(\frac{1}{2}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\rho_{01}f_\tau + \rho_{02}f_{\tau+\frac{1}{8}} + \rho_{03}f_{\tau+\frac{1}{4}} + \rho_{04}f_{\tau+\frac{3}{8}} + \\ \rho_{05}f_{\tau+\frac{1}{2}} + \rho_{06}f_{\tau+\frac{5}{8}} + \rho_{07}f_{\tau+\frac{3}{4}} + \rho_{08}f_{\tau+\frac{7}{8}} + \rho_{09}f_{\tau+1}\end{pmatrix} \\
y_{n+\frac{5}{8}} = y_n + \frac{5}{8}hy'_n + \frac{\left(\frac{5}{8}h\right)^2}{2!}y''_n + \frac{\left(\frac{5}{8}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\rho_{01}f_\tau + \rho_{02}f_{\tau+\frac{1}{8}} + \rho_{03}f_{\tau+\frac{1}{4}} + \rho_{04}f_{\tau+\frac{3}{8}} + \\ \rho_{05}f_{\tau+\frac{1}{2}} + \rho_{06}f_{\tau+\frac{5}{8}} + \rho_{07}f_{\tau+\frac{3}{4}} + \rho_{08}f_{\tau+\frac{7}{8}} + \rho_{09}f_{\tau+1}\end{pmatrix} \\
y_{n+\frac{3}{4}} = y_n + \frac{3}{4}hy'_n + \frac{\left(\frac{3}{4}h\right)^2}{2!}y''_n + \frac{\left(\frac{3}{4}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\rho_{01}f_\tau + \rho_{02}f_{\tau+\frac{1}{8}} + \rho_{03}f_{\tau+\frac{1}{4}} + \rho_{04}f_{\tau+\frac{3}{8}} + \\ \rho_{05}f_{\tau+\frac{1}{2}} + \rho_{06}f_{\tau+\frac{5}{8}} + \rho_{07}f_{\tau+\frac{3}{4}} + \rho_{08}f_{\tau+\frac{7}{8}} + \rho_{09}f_{\tau+1}\end{pmatrix} \\
y_{n+\frac{7}{8}} = y_n + \frac{7}{8}hy'_n + \frac{\left(\frac{7}{8}h\right)^2}{2!}y''_n + \frac{\left(\frac{7}{8}h\right)^3}{3!}y'''_n + h^4\begin{pmatrix}\rho_{01}f_\tau + \rho_{02}f_{\tau+\frac{1}{8}} + \rho_{03}f_{\tau+\frac{1}{4}} + \rho_{04}f_{\tau+\frac{3}{8}} + \\ \rho_{05}f_{\tau+\frac{1}{2}} + \rho_{06}f_{\tau+\frac{5}{8}} + \rho_{07}f_{\tau+\frac{3}{4}} + \rho_{08}f_{\tau+\frac{7}{8}} + \rho_{09}f_{\tau+1}\end{pmatrix} \\
y_{n+1} = y_n + hy'_n + \frac{(h)^2}{2!}y''_n + \frac{(h)^3}{3!}y'''_n + h^4\begin{pmatrix}\rho_{01}f_\tau + \rho_{02}f_{\tau+\frac{1}{8}} + \rho_{03}f_{\tau+\frac{1}{4}} + \rho_{04}f_{\tau+\frac{3}{8}} + \\ \rho_{05}f_{\tau+\frac{1}{2}} + \rho_{06}f_{\tau+\frac{5}{8}} + \rho_{07}f_{\tau+\frac{3}{4}} + \rho_{08}f_{\tau+\frac{7}{8}} + \rho_{09}f_{\tau+1}\end{pmatrix}
\end{array}\right\} \qquad (6)$$

Similarly**,** the linear block approach (4) is expanded to yield the higher derivatives as

$$
\left.\begin{aligned}
y'_{n+\frac{1}{8}} &= y'_n+\frac{1}{8}hy''_n+\frac{\left(\frac{1}{8}h\right)^2}{2!}y'''_n+h^3\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'_{n+\frac{1}{4}} &= y'_n+\frac{1}{4}hy''_n+\frac{\left(\frac{1}{4}h\right)^2}{2!}y'''_n+h^3\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'_{n+\frac{3}{8}} &= y'_n+\frac{3}{8}hy''_n+\frac{\left(\frac{3}{8}h\right)^2}{2!}y'''_n+h^3\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'_{n+\frac{1}{2}} &= y'_n+\frac{1}{2}hy''_n+\frac{\left(\frac{1}{2}h\right)^2}{2!}y'''_n+h^3\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'_{n+\frac{5}{8}} &= y'_n+\frac{5}{8}hy''_n+\frac{\left(\frac{5}{8}h\right)^2}{2!}y'''_n+h^3\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'_{n+\frac{3}{4}} &= y'_n+\frac{3}{4}hy''_n+\frac{\left(\frac{3}{4}h\right)^2}{2!}y'''_n+h^3\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'_{n+\frac{7}{8}} &= y'_n+\frac{7}{8}hy''_n+\frac{\left(\frac{7}{8}h\right)^2}{2!}y'''_n+h^3\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'_{n+1} &= y'_n+hy''_n+\frac{(h)^2}{2!}y'''_n+h^3\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)
\end{aligned}\right\} \quad (7)
$$

$$
\left.\begin{aligned}
y''_{n+\frac{1}{8}} &= y''_n+\frac{1}{8}hy'''_n+h^2\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y''_{n+\frac{1}{4}} &= y''_n+\frac{1}{4}hy'''_n+h^2\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y''_{n+\frac{3}{8}} &= y''_n+\frac{3}{8}hy'''_n+h^2\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y''_{n+\frac{1}{2}} &= y''_n+\frac{1}{2}hy'''_n+h^2\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y''_{n+\frac{5}{8}} &= y''_n+\frac{5}{8}hy'''_n+h^2\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y''_{n+\frac{3}{4}} &= y''_n+\frac{3}{4}hy'''_n+h^2\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y''_{n+\frac{7}{8}} &= y''_n+\frac{7}{8}hy'''_n+h^2\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y''_{n+1} &= y''_n+hy'''_n+h^2\left(\vartheta_{11}f_\tau+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)
\end{aligned}\right\} \quad (8)
$$

$$\left.\begin{aligned}
y'''_{n+\frac{1}{8}} &= y'''_{n}+h\left(\vartheta_{11}f_{\tau}+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'''_{n+\frac{1}{4}} &= y'''_{n}+h\left(\vartheta_{11}f_{\tau}+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'''_{n+\frac{3}{8}} &= y'''_{n}+h\left(\vartheta_{11}f_{\tau}+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'''_{n+\frac{1}{2}} &= y'''_{n}+h\left(\vartheta_{11}f_{\tau}+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'''_{n+\frac{5}{8}} &= y'''_{n}+h\left(\vartheta_{11}f_{\tau}+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'''_{n+\frac{3}{4}} &= y'''_{n}+h\left(\vartheta_{11}f_{\tau}+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'''_{n+\frac{7}{8}} &= y'''_{n}+h\left(\vartheta_{11}f_{\tau}+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)\\
y'''_{n+1} &= y'''_{n}+h\left(\vartheta_{11}f_{\tau}+\vartheta_{12}f_{\tau+\frac{1}{8}}+\vartheta_{13}f_{\tau+\frac{1}{4}}+\vartheta_{14}f_{\tau+\frac{3}{8}}+\vartheta_{15}f_{\tau+\frac{1}{2}}+\vartheta_{16}f_{\tau+\frac{5}{8}}+\vartheta_{17}f_{\tau+\frac{3}{4}}+\vartheta_{18}f_{\tau+\frac{7}{8}}+\vartheta_{19}f_{\tau+1}\right)
\end{aligned}\right\} \quad (9)$$

Therefore, in order to obtain the unknown coefficients of ρ, we consider $\rho_{\zeta j} = \mathrm{X}^{-1}\mathrm{A}$
where

$$y_{n+\frac{1}{8}}\begin{pmatrix}\rho_{01}\\\rho_{02}\\\rho_{03}\\\rho_{04}\\\rho_{05}\\\rho_{06}\\\rho_{07}\\\rho_{08}\\\rho_{09}\end{pmatrix}=\begin{pmatrix}\frac{24396497}{3923981107200}\\\frac{1520909}{163499212800}\\-\frac{13220819}{980995276800}\\\frac{8390797}{490497638400}\\-\frac{2050007}{130799370240}\\\frac{4854761}{490497638400}\\-\frac{364589}{89181388800}\\\frac{162689}{163499212800}\\-\frac{425111}{3923981107200}\end{pmatrix};\ y_{n+\frac{1}{4}}\begin{pmatrix}\rho_{01}\\\rho_{02}\\\rho_{03}\\\rho_{04}\\\rho_{05}\\\rho_{06}\\\rho_{07}\\\rho_{08}\\\rho_{09}\end{pmatrix}=\begin{pmatrix}\frac{1035731}{15328051200}\\\frac{169969}{958003200}\\-\frac{37379}{182476800}\\\frac{245837}{958003200}\\-\frac{357779}{1532805120}\\\frac{46873}{319334400}\\-\frac{231691}{3832012800}\\\frac{14071}{958003200}\\-\frac{8159}{5109350400}\end{pmatrix};\ y_{n+\frac{3}{8}},\begin{pmatrix}\rho_{01}\\\rho_{02}\\\rho_{03}\\\rho_{04}\\\rho_{05}\\\rho_{06}\\\rho_{07}\\\rho_{08}\\\rho_{09}\end{pmatrix}=\begin{pmatrix}\frac{4104531}{16148070400}\\\frac{156411}{183500800}\\-\frac{3119229}{4037017600}\\\frac{59337}{57671680}\\-\frac{1516887}{1614807040}\\\frac{1194183}{2018508800}\\-\frac{984537}{4037017600}\\\frac{17091}{288358400}\\-\frac{20817}{3229614080}\end{pmatrix};\ y_{n+\frac{1}{2}}\begin{pmatrix}\rho_{01}\\\rho_{02}\\\rho_{03}\\\rho_{04}\\\rho_{05}\\\rho_{06}\\\rho_{07}\\\rho_{08}\\\rho_{09}\end{pmatrix}=\begin{pmatrix}\frac{9521}{14968800}\\\frac{1511}{623700}\\-\frac{5239}{2993760}\\\frac{5011}{1871100}\\-\frac{25}{10368}\\\frac{2843}{1871100}\\-\frac{9379}{14968800}\\\frac{19}{124740}\\-\frac{31}{1871100}\end{pmatrix};$$

$$y_{n+\frac{5}{8}}\begin{pmatrix}\rho_{01}\\\rho_{02}\\\rho_{03}\\\rho_{04}\\\rho_{05}\\\rho_{06}\\\rho_{07}\\\rho_{08}\\\rho_{09}\end{pmatrix}=\begin{pmatrix}\frac{201421625}{156959244288}\\\frac{103514375}{19619905536}\\-\frac{40900625}{13079937024}\\\frac{111998125}{19619905536}\\-\frac{383340625}{78479622144}\\\frac{2901125}{934281216}\\-\frac{50269375}{39239811072}\\\frac{79375}{254803968}\\-\frac{1773125}{52319748096}\end{pmatrix};\ y_{n+\frac{3}{4}}\begin{pmatrix}\rho_{01}\\\rho_{02}\\\rho_{03}\\\rho_{04}\\\rho_{05}\\\rho_{06}\\\rho_{07}\\\rho_{08}\\\rho_{09}\end{pmatrix}=\begin{pmatrix}\frac{142929}{63078400}\\\frac{38637}{3942400}\\-\frac{77193}{15769600}\\\frac{42057}{3942400}\\-\frac{53217}{6307840}\\\frac{21951}{3942400}\\-\frac{5139}{2252800}\\\frac{2187}{3942400}\\-\frac{3807}{63078400}\end{pmatrix};\ y_{n+\frac{7}{8}}\begin{pmatrix}\rho_{01}\\\rho_{02}\\\rho_{03}\\\rho_{04}\\\rho_{05}\\\rho_{06}\\\rho_{07}\\\rho_{08}\\\rho_{09}\end{pmatrix}=\begin{pmatrix}\frac{2048300303}{560568729600}\\\frac{382678583}{23357030400}\\-\frac{89766187}{12740198400}\\\frac{1268609167}{70071091200}\\-\frac{244827569}{18685624320}\\\frac{650716619}{70071091200}\\-\frac{512596693}{140142182400}\\\frac{21001547}{23357030400}\\-\frac{54841241}{560568729600}\end{pmatrix};\ y_{n+1}\begin{pmatrix}\rho_{01}\\\rho_{02}\\\rho_{03}\\\rho_{04}\\\rho_{05}\\\rho_{06}\\\rho_{07}\\\rho_{08}\\\rho_{09}\end{pmatrix}=\begin{pmatrix}\frac{2581}{467775}\\\frac{11888}{467775}\\-\frac{1492}{155925}\\\frac{2672}{93555}\\-\frac{1769}{93555}\\\frac{208}{14175}\\-\frac{2468}{467775}\\\frac{656}{467775}\\-\frac{37}{249480}\end{pmatrix}$$

Similarly, the unknown coefficients of ϑ is given by $\vartheta_{\zeta\, jq} = \mathrm{X}^{-1}\mathrm{N}$

where

$$y'_{n+\frac{1}{8}}\begin{pmatrix}\vartheta_{11}\\ \vartheta_{12}\\ \vartheta_{13}\\ \vartheta_{14}\\ \vartheta_{15}\\ \vartheta_{16}\\ \vartheta_{17}\\ \vartheta_{18}\\ \vartheta_{19}\end{pmatrix}=\begin{pmatrix}\frac{517129}{2919628800}\\ \frac{1129981}{3406233600}\\ -\frac{1871827}{4087480320}\\ \frac{5887073}{10218700800}\\ -\frac{716363}{1362493440}\\ \frac{3385541}{10218700800}\\ -\frac{2792861}{20437401600}\\ \frac{22637}{681246720}\\ -\frac{36943}{10218700800}\end{pmatrix},\ y'_{n+\frac{1}{4}}\begin{pmatrix}\vartheta_{11}\\ \vartheta_{12}\\ \vartheta_{13}\\ \vartheta_{14}\\ \vartheta_{15}\\ \vartheta_{16}\\ \vartheta_{17}\\ \vartheta_{18}\\ \vartheta_{19}\end{pmatrix}=\begin{pmatrix}\frac{286967}{319334400}\\ \frac{32543}{11404800}\\ -\frac{22063}{7603200}\\ \frac{58657}{15966720}\\ -\frac{21359}{6386688}\\ \frac{55969}{26611200}\\ -\frac{138317}{159667200}\\ \frac{16799}{79833600}\\ -\frac{487}{21288960}\end{pmatrix},\ y'_{n+\frac{3}{8}}\begin{pmatrix}\vartheta_{11}\\ \vartheta_{12}\\ \vartheta_{13}\\ \vartheta_{14}\\ \vartheta_{15}\\ \vartheta_{16}\\ \vartheta_{17}\\ \vartheta_{18}\\ \vartheta_{19}\end{pmatrix}=\begin{pmatrix}\frac{68769}{31539200}\\ \frac{1067877}{126156800}\\ -\frac{1563651}{252313600}\\ \frac{32841}{3604480}\\ -\frac{419931}{50462720}\\ \frac{661959}{126156800}\\ -\frac{546129}{252313600}\\ \frac{66393}{126156800}\\ -\frac{2889}{50462720}\end{pmatrix},\ y'_{n+\frac{1}{2}}\begin{pmatrix}\vartheta_{11}\\ \vartheta_{12}\\ \vartheta_{13}\\ \vartheta_{14}\\ \vartheta_{15}\\ \vartheta_{16}\\ \vartheta_{17}\\ \vartheta_{18}\\ \vartheta_{19}\end{pmatrix}=\begin{pmatrix}\frac{80293}{19958400}\\ \frac{3571}{207900}\\ -\frac{4703}{498960}\\ \frac{11213}{623700}\\ -\frac{37}{2376}\\ \frac{6131}{623700}\\ -\frac{10117}{2494800}\\ \frac{41}{41580}\\ -\frac{2141}{19958400}\end{pmatrix},$$

$$y'_{n+\frac{5}{8}}\begin{pmatrix}\vartheta_{11}\\ \vartheta_{12}\\ \vartheta_{13}\\ \vartheta_{14}\\ \vartheta_{15}\\ \vartheta_{16}\\ \vartheta_{17}\\ \vartheta_{18}\\ \vartheta_{19}\end{pmatrix}=\begin{pmatrix}\frac{5253125}{817496064}\\ \frac{11851375}{408748032}\\ -\frac{3432125}{272498688}\\ \frac{12768625}{408748032}\\ -\frac{19680625}{817496064}\\ \frac{307625}{19464192}\\ -\frac{5327125}{817496064}\\ \frac{647875}{408748032}\\ -\frac{5875}{34062336}\end{pmatrix},\ y'_{n+\frac{3}{4}}\begin{pmatrix}\vartheta_{11}\\ \vartheta_{12}\\ \vartheta_{13}\\ \vartheta_{14}\\ \vartheta_{15}\\ \vartheta_{16}\\ \vartheta_{17}\\ \vartheta_{18}\\ \vartheta_{19}\end{pmatrix}=\begin{pmatrix}\frac{37017}{3942400}\\ \frac{6183}{140800}\\ -\frac{6183}{394240}\\ \frac{48141}{985600}\\ -\frac{12933}{394240}\\ \frac{23787}{985600}\\ -\frac{2691}{281600}\\ \frac{459}{197120}\\ -\frac{999}{3942400}\end{pmatrix},\ y'_{n+\frac{7}{8}}\begin{pmatrix}\vartheta_{11}\\ \vartheta_{12}\\ \vartheta_{13}\\ \vartheta_{14}\\ \vartheta_{15}\\ \vartheta_{16}\\ \vartheta_{17}\\ \vartheta_{18}\\ \vartheta_{19}\end{pmatrix}=\begin{pmatrix}\frac{18850937}{1459814400}\\ \frac{30139753}{486604800}\\ -\frac{54639557}{2919628800}\\ \frac{20689417}{291962880}\\ -\frac{1630279}{38928384}\\ \frac{52421033}{1459814400}\\ -\frac{35782103}{2919628800}\\ \frac{1570597}{486604800}\\ -\frac{40817}{116785152}\end{pmatrix},\ y'_{n+1}\begin{pmatrix}\vartheta_{11}\\ \vartheta_{12}\\ \vartheta_{13}\\ \vartheta_{14}\\ \vartheta_{15}\\ \vartheta_{16}\\ \vartheta_{17}\\ \vartheta_{18}\\ \vartheta_{19}\end{pmatrix}=\begin{pmatrix}\frac{3029}{178200}\\ \frac{12952}{155925}\\ -\frac{1126}{51975}\\ \frac{3032}{31185}\\ -\frac{3191}{62370}\\ \frac{2648}{51975}\\ -\frac{2102}{155925}\\ \frac{808}{155925}\\ -\frac{37}{83160}\end{pmatrix},$$

$$y''_{n+\frac{1}{8}}\begin{pmatrix}\vartheta_{21}\\ \vartheta_{22}\\ \vartheta_{23}\\ \vartheta_{24}\\ \vartheta_{25}\\ \vartheta_{26}\\ \vartheta_{27}\\ \vartheta_{28}\\ \vartheta_{29}\end{pmatrix}=\begin{pmatrix}\frac{324901}{92897280}\\ \frac{8183}{921600}\\ -\frac{653203}{58060800}\\ \frac{50689}{3628800}\\ -\frac{196277}{15482880}\\ \frac{92473}{11612160}\\ -\frac{95167}{29030400}\\ \frac{7703}{9676800}\\ -\frac{5741}{66355200}\end{pmatrix},\ y''_{n+\frac{1}{4}}\begin{pmatrix}\vartheta_{21}\\ \vartheta_{22}\\ \vartheta_{23}\\ \vartheta_{24}\\ \vartheta_{25}\\ \vartheta_{26}\\ \vartheta_{27}\\ \vartheta_{28}\\ \vartheta_{29}\end{pmatrix}=\begin{pmatrix}\frac{58193}{7257600}\\ \frac{3673}{113400}\\ -\frac{81}{3200}\\ \frac{7729}{226800}\\ -\frac{22703}{725760}\\ \frac{373}{18900}\\ -\frac{14773}{1814400}\\ \frac{449}{226800}\\ -\frac{521}{2419200}\end{pmatrix},\ y''_{n+\frac{3}{8}}\begin{pmatrix}\vartheta_{21}\\ \vartheta_{22}\\ \vartheta_{23}\\ \vartheta_{24}\\ \vartheta_{25}\\ \vartheta_{26}\\ \vartheta_{27}\\ \vartheta_{28}\\ \vartheta_{29}\end{pmatrix}=\begin{pmatrix}\frac{71661}{5734400}\\ \frac{1467}{25600}\\ -\frac{4707}{179200}\\ \frac{225}{4096}\\ -\frac{28143}{573440}\\ \frac{11079}{358400}\\ -\frac{9141}{716800}\\ \frac{2223}{716800}\\ -\frac{387}{1146880}\end{pmatrix},\ y''_{n+\frac{1}{2}}\begin{pmatrix}\vartheta_{21}\\ \vartheta_{22}\\ \vartheta_{23}\\ \vartheta_{24}\\ \vartheta_{25}\\ \vartheta_{26}\\ \vartheta_{27}\\ \vartheta_{28}\\ \vartheta_{29}\end{pmatrix}=\begin{pmatrix}\frac{7703}{453600}\\ \frac{388}{4725}\\ -\frac{29}{222}\\ \frac{1252}{14175}\\ -\frac{47}{720}\\ \frac{596}{14175}\\ -\frac{493}{28350}\\ \frac{4}{945}\\ -\frac{209}{453600}\end{pmatrix},$$

$$y''_{n+\frac{5}{8}}\begin{pmatrix}\vartheta_{21}\\\vartheta_{22}\\\vartheta_{23}\\\vartheta_{24}\\\vartheta_{25}\\\vartheta_{26}\\\vartheta_{27}\\\vartheta_{28}\\\vartheta_{29}\end{pmatrix}=\begin{pmatrix}\frac{56975}{2654208}\\\frac{248375}{2322432}\\-\frac{19375}{774144}\\\frac{143375}{1161216}\\-\frac{641875}{9289728}\\\frac{225}{4096}\\-\frac{12875}{580608}\\\frac{3125}{580608}\\-\frac{3625}{6193152}\end{pmatrix},\ y''_{n+\frac{3}{4}}\begin{pmatrix}\vartheta_{21}\\\vartheta_{22}\\\vartheta_{23}\\\vartheta_{24}\\\vartheta_{25}\\\vartheta_{26}\\\vartheta_{27}\\\vartheta_{28}\\\vartheta_{29}\end{pmatrix}=\begin{pmatrix}\frac{93}{3584}\\\frac{369}{2800}\\-\frac{549}{22400}\\\frac{111}{700}\\-\frac{639}{8960}\\\frac{9}{112}\\-\frac{81}{3200}\\\frac{9}{1400}\\-\frac{9}{12800}\end{pmatrix},\ y''_{n+\frac{7}{8}}\begin{pmatrix}\vartheta_{21}\\\vartheta_{22}\\\vartheta_{23}\\\vartheta_{24}\\\vartheta_{25}\\\vartheta_{26}\\\vartheta_{27}\\\vartheta_{28}\\\vartheta_{29}\end{pmatrix}=\begin{pmatrix}\frac{2019731}{66355200}\\\frac{216433}{1382400}\\-\frac{98441}{4147200}\\\frac{1601467}{8294400}\\-\frac{160867}{2211840}\\\frac{55223}{518400}\\-\frac{127253}{8294400}\\\frac{8183}{921600}\\-\frac{57281}{66355200}\end{pmatrix},\ y''_{n+1}\begin{pmatrix}\vartheta_{21}\\\vartheta_{22}\\\vartheta_{23}\\\vartheta_{24}\\\vartheta_{25}\\\vartheta_{26}\\\vartheta_{27}\\\vartheta_{28}\\\vartheta_{29}\end{pmatrix}=\begin{pmatrix}\frac{989}{28350}\\\frac{368}{2025}\\-\frac{116}{4725}\\\frac{656}{2835}\\-\frac{227}{2835}\\\frac{656}{4725}\\-\frac{116}{14175}\\\frac{368}{14175}\\0\end{pmatrix},$$

$$y'''_{n+\frac{1}{8}}\begin{pmatrix}\vartheta_{31}\\\vartheta_{32}\\\vartheta_{33}\\\vartheta_{34}\\\vartheta_{35}\\\vartheta_{26}\\\vartheta_{37}\\\vartheta_{38}\\\vartheta_{39}\end{pmatrix}=\begin{pmatrix}\frac{1070017}{29030400}\\\frac{2233547}{14515200}\\-\frac{2302297}{14515200}\\\frac{2797679}{14515200}\\-\frac{31457}{181440}\\\frac{1573169}{14515200}\\-\frac{645607}{14515200}\\\frac{156437}{14515200}\\-\frac{33953}{29030400}\end{pmatrix},\ y'''_{n+\frac{1}{4}}\begin{pmatrix}\vartheta_{31}\\\vartheta_{32}\\\vartheta_{33}\\\vartheta_{34}\\\vartheta_{35}\\\vartheta_{26}\\\vartheta_{37}\\\vartheta_{38}\\\vartheta_{39}\end{pmatrix}=\begin{pmatrix}\frac{32377}{907200}\\\frac{22823}{113400}\\-\frac{21247}{453600}\\\frac{15011}{113400}\\-\frac{2903}{22680}\\\frac{9341}{113400}\\-\frac{15577}{453600}\\\frac{953}{113400}\\-\frac{119}{129600}\end{pmatrix},\ y'''_{n+\frac{3}{8}}\begin{pmatrix}\vartheta_{31}\\\vartheta_{32}\\\vartheta_{33}\\\vartheta_{34}\\\vartheta_{35}\\\vartheta_{26}\\\vartheta_{37}\\\vartheta_{38}\\\vartheta_{39}\end{pmatrix}=\begin{pmatrix}\frac{12881}{358400}\\\frac{35451}{179200}\\\frac{1719}{179200}\\\frac{39967}{179200}\\-\frac{351}{2240}\\\frac{17217}{179200}\\-\frac{7031}{179200}\\\frac{243}{25600}\\-\frac{369}{358400}\end{pmatrix},\ y'''_{n+\frac{1}{2}}\begin{pmatrix}\vartheta_{31}\\\vartheta_{32}\\\vartheta_{33}\\\vartheta_{34}\\\vartheta_{35}\\\vartheta_{26}\\\vartheta_{37}\\\vartheta_{38}\\\vartheta_{39}\end{pmatrix}=\begin{pmatrix}\frac{4063}{113400}\\\frac{2822}{14175}\\\frac{61}{28350}\\\frac{4094}{14175}\\-\frac{227}{2835}\\\frac{1154}{14175}\\-\frac{989}{28350}\\\frac{122}{14175}\\-\frac{107}{113400}\end{pmatrix},$$

$$y'''_{n+\frac{5}{8}}\begin{pmatrix}\vartheta_{31}\\\vartheta_{32}\\\vartheta_{33}\\\vartheta_{34}\\\vartheta_{35}\\\vartheta_{26}\\\vartheta_{37}\\\vartheta_{38}\\\vartheta_{39}\end{pmatrix}=\begin{pmatrix}\frac{41705}{1161216}\\\frac{115075}{580608}\\-\frac{3775}{580608}\\\frac{159175}{580608}\\-\frac{125}{36288}\\\frac{85465}{580608}\\-\frac{24575}{580608}\\\frac{5725}{580608}\\-\frac{175}{165888}\end{pmatrix},\ y'''_{n+\frac{3}{4}}\begin{pmatrix}\vartheta_{31}\\\vartheta_{32}\\\vartheta_{33}\\\vartheta_{34}\\\vartheta_{35}\\\vartheta_{26}\\\vartheta_{37}\\\vartheta_{38}\\\vartheta_{39}\end{pmatrix}=\begin{pmatrix}\frac{401}{11200}\\\frac{279}{1400}\\\frac{9}{5600}\\\frac{403}{1400}\\-\frac{9}{280}\\\frac{333}{1400}\\\frac{79}{5600}\\\frac{9}{1400}\\-\frac{9}{11200}\end{pmatrix},\ y'''_{n+\frac{7}{8}}\begin{pmatrix}\vartheta_{31}\\\vartheta_{32}\\\vartheta_{33}\\\vartheta_{34}\\\vartheta_{35}\\\vartheta_{26}\\\vartheta_{37}\\\vartheta_{38}\\\vartheta_{39}\end{pmatrix}=\begin{pmatrix}\frac{149527}{4147200}\\\frac{408317}{2073600}\\\frac{24353}{2073600}\\\frac{542969}{2073600}\\\frac{343}{25920}\\\frac{368039}{2073600}\\\frac{261023}{2073600}\\\frac{111587}{2073600}\\-\frac{8183}{4147200}\end{pmatrix},\ y'''_{n+1}\begin{pmatrix}\vartheta_{31}\\\vartheta_{32}\\\vartheta_{33}\\\vartheta_{34}\\\vartheta_{35}\\\vartheta_{26}\\\vartheta_{37}\\\vartheta_{38}\\\vartheta_{39}\end{pmatrix}=\begin{pmatrix}\frac{989}{28350}\\\frac{2944}{14175}\\-\frac{464}{14175}\\\frac{5248}{14175}\\-\frac{454}{2835}\\\frac{5248}{14175}\\-\frac{464}{14175}\\\frac{2944}{14175}\\-\frac{989}{28350}\end{pmatrix}$$

3. Evaluation of the New Schemes

This section examines the new schemes according to numerical analysis, viz., order and error constant, consistency, zero stability, convergence, and region of absolute stability.

3.1. Order and Error Constant of the new scheme

We consider the linear operator $L[y(t_n);h]$ with the corollary 2 and 3 below to determining the order and error constant of the new method.

Corollary 2

The linear operator $L[y(t_n);h]$ associate with the local truncation error of the new method is $C_{07}h^{07}y^{07}(t_n)+0(h^{11})$.

Proof

According to (Kwari *et al*. (2023)), the linear difference operators associated with the new method are given by

$$\left.\begin{aligned}
L[y(t_n);h]&=y\left(t_n+\frac{1}{8}h\right)-\left(\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\right)\\
L[y(t_n);h]&=y\left(t_n+\frac{1}{4}h\right)-\left(\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\right)\\
L[y(t_n);h]&=y\left(t_n+\frac{3}{8}h\right)-\left(\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\right)\\
L[y(t_n);h]&=y\left(t_n+\frac{1}{2}h\right)-\left(\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\right)\\
L[y(t_n);h]&=y\left(t_n+\frac{5}{8}h\right)-\left(\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\right)\\
L[y(t_n);h]&=y\left(t_n+\frac{3}{4}h\right)-\left(\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\right)\\
L[y(t_n);h]&=y\left(t_n+\frac{7}{8}h\right)-\left(\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\right)\\
L[y(t_n);h]&=y(t_n+h)-\left(\alpha_{\frac{1}{8}}\left(t_n+\frac{1}{8}h\right)+\alpha_{\frac{1}{4}}\left(t_n+\frac{1}{4}h\right)+\alpha_{\frac{3}{8}}\left(t_n+\frac{3}{8}h\right)+\alpha_{\frac{1}{2}}\left(x_n+\frac{1}{2}h\right)+h^4\sum_{i=0}^{\xi}\left(\beta_i(t)f_{n+i}+\beta_{\xi}(t)f_{n+\xi}\right)\right)
\end{aligned}\right\} \quad (10)$$

Corollary 3

The local truncation error of the new scheme is assumed $y(t)$ to be sufficiently differentiable and expanding $y(t_n+qh)$ and $y(t_n+jh)$ about t_n using Taylor series to have

$$L_{\frac{1}{8}}[y(t_n);h]=\left(-1.0827\times10^{-10}\right),\ L_{\frac{1}{4}}[y(t_n);h]=\left(-1.6013\times10^{-09}\right),\ L_{\frac{3}{8}}[y(t_n);h]=\left(-6.45917\times10^{-09}\right),$$

$$L_{\frac{1}{2}}[y(t_n);h]=\left(-1.6595\times10^{-08}\right),\ L_{\frac{5}{8}}[y(t_n);h]=\left(-3.3929\times10^{-08}\right),\ L_{\frac{3}{4}}[y(t_n);h]=\left(-6.0416\times10^{-08}\right),$$

$$L_{\frac{7}{8}}[y(t_n);h]=\left(-9.6874\times10^{-08}\right),\ L_{1}[y(t_n);h]=\left(-1.3974\times10^{-07}\right)$$

Proof

Expand equation (10) using corollary 3 and then collect the like terms to the power of h gives

$$L_{\frac{1}{8}}[y(t_n);h] = (-1.0827\times10^{-10})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{\frac{1}{4}}[y(t_n);h] = (-1.6013\times10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{\frac{3}{8}}[y(t_n);h] = (-6.45917\times10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{11}),$$
$$L_{\frac{1}{2}}[y(t_n);h] = (-1.6595\times10^{-08})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{\frac{5}{8}}[y(t_n);h] = (-3.3929\times10^{-08})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{\frac{3}{4}}[y(t_n);h] = (-6.0416\times10^{-08})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{\frac{7}{8}}[y(t_n);h] = (-9.6874\times10^{-08})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$
$$L_{1}[y(t_n);h] = (-1.3974\times10^{-07})C_{07}h^{07}y^{07}(t_n) + 0(h^{11})$$

3.2. Consistency

According to Kuboye *et al*. (2022), a linear multistep technique is considered consistent if its order of convergence is $(p \geq 1)$, meaning it is greater than or equal to zero. Since the order are 5, our new schemes are therefore consistent.

3.3. Zero Stability

For every well-behaved initial value problem, a linear multistep method is considered zero-stable if

i. in the unit disk, , $|r| \leq 1$, are all of the roots of $\rho(r)$

ii. on the unit circle $(|r| \leq 1)$, all roots are simple Raymond *et al*. (2023)

Thus,

$$\rho(z) = z^8 - \frac{1522}{35}z^7 + \frac{118124}{105}z^6 - \frac{102528}{5}z^5 + 273664z^4 - 2654208z^3 + 17891328z^2 + 7549742z + 150994944 \tag{11}$$

Now set (11) equal to zero and solving for z gives $z = 1$, hence the method is zero stable.

3.4. Convergence

According to Wend (1969), the necessary and sufficient condition for a linear multistep to be convergent is that, it must be consistent and zero stable. Since the new scheme is consistent and zero stable, hence it is convergent.

3.5. Linear Stability

The region of absolute stability of new scheme is the set of complex values λh for which all solutions of the test problem $y'''' = -\lambda^4 y$ will remain bounded as $n \to \infty$.

The concept of A-stability according to Raymond *et al*. (2023) is discussed by applying the test equation

$$y^{(k)} = \lambda^{(k)}y \tag{12}$$

to yield

$$Y_m = \mu(z)Y_{m-1},\ z = \lambda h \tag{13}$$

where $\mu(z)$ is the amplification matrix of the form

$$\mu(z) = \left(\xi^0 - z\eta^{(0)} - z^4\eta^{(0)}\right)^{-1}\left(\xi^1 - z\eta^{(1)} - z^4\eta^{(1)}\right) \tag{14}$$

The matrix $\mu(z)$ has Eigen values $(0, 0, \cdots, \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function of the method is given by

$$\zeta = -\frac{\begin{pmatrix} 1315895675\,85z^{8} - 6937972638\,972z^{7} + 16884257919\,9360z^{6} - 32999408631\,74816z^{5} \\ + 4262645178\,4068096z^{4} - 4300268967\,04677888z^{3} - 2838814659\,35100288z^{2} - \\ 1227258005\,1546931200\,z + 2416373296\,7 \end{pmatrix}}{\begin{pmatrix} 8001504000\,z^{8} - 3479511168\,000z^{7} + 9001615792\,000z^{6} - 1640756404\,224000z^{5} + \\ 2189723590\,6560000z^{4} - 212\,3765592883\,20000z^{3} + 1431575325\,573120000^{2} - \\ 6040933241\,978880000z + 1208186648\,3957760000 \end{pmatrix}} \quad (15)$$

The boundary locus method is adopted in generating the stability polynomial of the hybrid method. The polynomial is

$$\left.\begin{aligned} \bar{h}(w) = &\left(-\frac{1}{24576}w^{5} + \frac{1}{286720}w^{6}\right)h^{6} + \left(-\frac{3}{143360}w^{6} - \frac{1019}{1290240}w^{5}\right)h^{5} + \left(-\frac{1849}{215040}w^{5} - \frac{71}{215040}w^{6}\right)h^{4} \\ &+ \left(\frac{1}{224}w^{5} - \frac{41}{14}w^{6}\right)h^{3} + \left(-\frac{95}{336}w^{5} - \frac{1}{336}w^{6}\right)h^{2} + \left(-\frac{3}{14}w^{5} - \frac{11}{14}w^{6}\right)h + w^{5} + w^{6} \end{aligned}\right\} \quad (16)$$

The polynomial is used to plot the region as

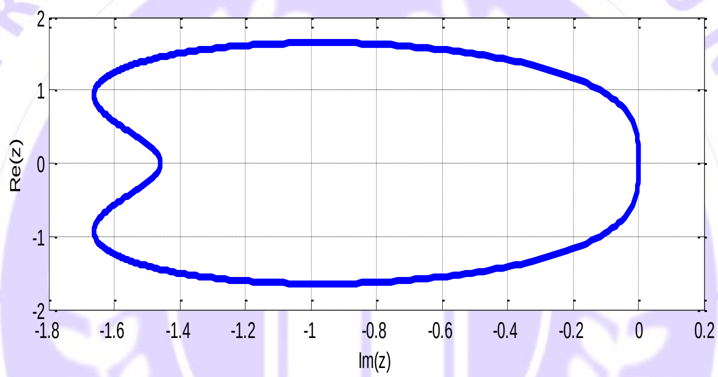

Figure. 1: Regions of absolute stability of the method.

4. Numerical Examples with Results

The accuracy of the multi order schemes has been tested on some second, third and fourth order oscillatory differential equations from linear to non-linear systems and physical problem. The new schemes were directly solved due to the experienced in conventional process when solving higher order oscillatory equation (1). The computation is carried out on Maple 18 software package.

The absolute error from the new method is compared with some existing methods where step-size h varies.

The following acronyms are used in the Tables 1 to Table 6 and Figure I.

ES: Exact Solution; NM: New Method; CS: Computed Solution; ENM: Error in New Method; EEM: Error in Existing Method [Agarwal *et al*. (2013)]; Error in [Agarwal *et al*. (2013)]; E[Kuboye *et al*. (2022)]: Error in [Kuboye *et al.* (2022)]; E[Adoghe *et al.* (2019)]: Error in [Adoghe *et al.* (2019)]; E[Areo and Rufai (2016)]: Error in [Areo and Rufai (2016)]; E[Olabode and Momoh (2016)]: Error in [Olabode and Momoh (2016)]; E[Jator and Li (2009)]: Error in [Jator and Li (2009)]; E[Omar and Abdelrahim (2016)]: Error in [Omar and Abdelrahim (2016)]; E[Kayode and Obaruha (2017)]: Error in [Kayode and Obaruha (2017)]; E[Adeyeye and Omar (2019)]: Error in [Adeyeye and Omar (2019)]; E[Fasasi *et al*. (2015)]: Error in [Fasasi *et al.* (2015)]; E[Areo and Omojola (2017)]: Error in [Areo and Omojola (2017)]; E[Sunday (2018)]: Error in [Sunday (2018)]; E[Sunday *et al*. (2019)]: Error in [Sunday *et al.* (2019)]; E[Adeyeye and Omar (2019)]: Error in [Adeyeye and Omar (2019)]; E[Allogmany *et al*. (2020)]: Error in [Allogmany *et al*. (2020)]; E[Kuboye and Omar (2015)]: Error in [Kuboye and Omar (2015)]; E[Ahamad and Charan (2019)]: Error in [Ahamad and Charan (2019)]; E[Ukpebor *et al*. (2020)]: Error in [Ukpebor *et al*. (2020)]; E[Atabo and Adee. (2021)]: Error in [Atabo and Adee. (2021)].

Example 1

Consider the highly stiff oscillatory third order problematic differential equation

$$y'''+y'=0,\ y(0)=0,\ y'(0)=1,\ y''(0)=2,\ h=0.1,\ 0.05 \tag{17}$$

with the exact solution given by

$$y(t)=2(1-\cos t)+\sin t \tag{18}$$

See: [Fasasi *et al*. (2015), Areo and Omojola (2017), Sunday (2018), Sunday *et al*. (2019)].

Table 1 Computation of NM with ES when solved example 1

T	**ES**	**CS**	**ES**	**CS**
	$h=0.1$		$h=0.05$	
0.1	0.10982508609077662011	0.10982508609077662012	0.05247864848074583570	0.05247864848074583567
0.2	0.23853617511257795326	0.23853617511257795321	0.10982508609077662011	0.10982508609077662012
0.3	0.38484722841012753581	0.38484722841012753581	0.17189597660151464800	0.17189597660151464803
0.4	0.54729635430288032607	0.54729635430288032606	0.23853617511257795326	0.23853617511257795321
0.5	0.72426041482345756807	0.72426041482345756802	0.30957911583323336130	0.30957911583323336130
0.6	0.91397124357567876270	0.91397124357567876268	0.38484722841012753581	0.38484722841012753581
0.7	1.11453331266871420120	1.11453331266871420110	0.46415238176069350909	0.46415238176069350910
0.8	1.32394267220519191980	1.32394267220519191970	0.54729635430288032607	0.54729635430288032603
0.9	1.54010697308615447550	1.54010697308615447540	0.63407132940587636712	0.63407132940587636703
1.0	1.76086637307161707180	1.76086637307161707170	0.72426041482345756807	0.72426041482345756816

Table 2 Absolute Error of NM with [Agarwal *et al*. (2019), Fasasi *et al*. (2015), Areo and Omojola (2017), Sunday (2018), Sunday *et al*. (2019)] for example 1

ENM		**EEM** $h=0.1$				**EEM** $h=0.05$	
$h=0.1$	$h=0.05$	**E[**Agarwal *et al*. (2019)**]**	**E[**Areo and Omojola (2017)**]**	**E[**Sunday (2018)**]**	**E[**Sunday *et al*. (2019)**]**	**E[**Agarwal *et al*. (2019)**]**	**E[**Fasasi *et al*. (2015)**]**
1.0000(-20)	3.0000(-20)	1.9345(-14)	1.1177e-10	3.7470e-16	2.4980e-16	1.1796(-16)	2.0650(-14)
5.0000(-20)	1.0000(-20)	7.1831(-14)	9.3348e-10	8.3267e-16	4.1633e-16	2.2204(-16)	1.9500(-11)
0.0000(00)	3.0000(-20)	1.8218(-13)	3.2775e-09	1.3878e-15	8.3267e-16	8.3267(-17)	8.0940(-11)
1.0000(-20)	5.0000(-20)	3.6803(-13)	8.0524e-09	1.4433e-15	1.4433e-16	8.0491(-16)	1.9640(-10)
5.0000(-20)	0.0000(00)	6.5725(-13)	1.6249e-08	1.5543e-15	4.4409e-16	9.9920(-16)	3.7020(-10)
2.0000(-20)	0.0000(00)	1.0689(-12)	2.8912e-08	1.9986e-15	1.1102e-16	2.1649(-15)	-
1.0000(-19)	1.0000(-20)	1.6320(-12)	4.7125e-08	2.8866e-15	4.4409e-19	3.4972(-15)	-
1.0000(-19)	4.0000(-20)	2.3652(-12)	7.1985e-08	4.4409e-15	1.3323e-15	5.9952(-15)	-
1.0000(-19)	9.0000(-20)	3.2969(-12)	1.0458e-07	3.5527e-15	4.4409e-15	8.5487(-15)	-
1.0000(-19)	9.0000(-20)	4.4442(-12)	1.4596e-07	5.3291e-15	2.2204e-15	1.0991(-14)	-

Example 2: Consider the fourth order oscillatory differential equation

$$y^{iv}=-y'',\ y(0)=0,\ y'(0)=\frac{-1.1}{72-50\pi},\ y''(0)=\frac{1}{144-50\pi},\ y'''(0)=\frac{1.2}{144-100\pi} \tag{19}$$

with the exact solution,

$$y(t)=\frac{1-t\cos t-1.2\sin t}{144-100\pi} \tag{20}$$

where $t\in\left[0,\frac{\pi}{2}\right]$. This problem was also solved by [Adoghe and Omole (2019), Adeyeye and Omar (2019), Allogmany *et al*. (2020), Kuboye and Omar (2015)].

Table 3 Computation of NM with ES for solved example 2

t	**ES** $h = 0.103125$	**CS**	**ENM** $h = 0.103125$	$h = 0.05$	$h = 0.01$	$h = 0.1$
1	0.00129983365565590794	0.00129983365565590794	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
2	0.00252989368302292067	0.00252989368302292067	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
3	0.00368354497764325043	0.00368354497764325043	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
4	0.00475496430413771076	0.00475496430413771076	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
5	0.00573920217008546594	0.00573920217008546594	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
6	0.00663223541608418253	0.00663223541608418253	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
7	0.00743100996320309685	0.00743100996320309685	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
8	0.00813347326362259984	0.00813347326362259984	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
9	0.00873859610966112687	0.00873859610966112687	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
10	0.00924638356946093719	0.00924638356946093719	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)

Table 4: Absolute error of NM with [Adoghe and Omole (2019), Adeyeye and Omar (2019), Allogmany *et al*. (2020), Kuboye and Omar (2015), Ahamad and Charan (2019)] for example 2

t	**EEM E[**Adeyeye and Umar(2019)**]** $h = 0.01$	$h = 0.025$	$h = 0.05$	$h = 0.1$	**E[**Adoghe and Omole (2019)**]** $h = 0.103125$	**E[**Allogmany *et al*. (2020)**]** $h = 0.003125$	**E[**Kuboye and Omar (2015)**]** $h = 0.01$	**E[**Ahamad and Charan (2019)**]** $h = 0.01$
1	6.5021(-19)	6.5052(-19)	5.4210(-18)	4.6295(-16)	2.1149(-18)	2.1656(-19)	5.4210(-20)	5.4203(-20)
2	8.6736(-19)	9.3368(-19)	6.2884(-17)	6.9966(-15)	1.0576(-17)	2.1629(-18)	5.4210(-20)	5.4196(-20)
3	8.6736(-19)	1.7347(-18)	2.3289(-16)	2.8003(-14)	1.2683(-17)	6.9136(-18)	2.7105(-19)	2.7095(-19)
4	2.6021(-18)	6.9389(-18)	6.3578(-16)	7.1500(-14)	2.1963(-17)	1.1223(-17)	1.0842(-19)	0.0000(00)
5	2.6021(-18)	1.9082(-17)	1.4554(-15)	1.4503(-13)	2.5623(-17)	1.4662(-17)	3.2526(-19)	3.2505(-19)
6	2.6021(-18)	3.9899(-17)	2.8970(-15)	2.5942(-13)	3.7297(-17)	4.3088(-18)	3.2526(-19)	3.2501(-19)
7	3.4695(-18)	7.3726(-17)	5.2623(-15)	4.3649(-13)	4.4098(-17)	8.6108(-18)	-	-
8	1.7347(-18)	1.2837(-16)	8.8714(-15)	8.8714(-13)	5.9762(-17)	1.8931(17)	1.7347(18)	0.0000(00)
9	1.7347(-18)	2.0817(-16)	1.4107(-14)	1.0825(-12)	7.1313(-17)	2.9239(-17)	4.3368(-18)	2.1659(-19)
10	1.7347(-18)	3.1919(-16)	2.1415(-14)	1.5981(-12)	9.2590(-17)	-	8.4568(-18)	0.0000(00)

Example 3: Consider the fourth order oscillatory differential equation

$$y^{iv} = t,\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 1,\ y'''(0) = 0 \tag{21}$$

is a fourth order problem with step size $h = 0.1, 0.01, 0.003125, 0.103125$ Solved by [Kuboye *et al.* (2022), Adoghe and Omole (2019), Ahamad and Charan (2019), Ukpebor *et al*. (2020), Atabo and Adee (2021)] where the exact solution is

$$y(t) = \frac{t^5}{120t\pi} + t \tag{22}$$

Table 5: Computation of NM with ES for solved example 3

T	**ES** $h = 0.103125$	**CS**	**ENM** $h = 0.103125$	$h = 0.003125$	$h = 0.01$	$h = 0.1$
1	0.00312500000000248353	0.00312500000000248353	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
2	0.00625000000007947286	0.00625000000007947286	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
3	0.00937500000060349703	0.00937500000060349703	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
4	0.01250000000254313151	0.01250000000254313151	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
5	0.01562500000776102146	0.01562500000776102146	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
6	0.01875000001931190491	0.01875000001931190491	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
7	0.02187500004174063603	0.02187500004174063603	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)

8	0.02500000008138020833	0.02500000008138020833	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
9	0.02812500014664977789	0.02812500014664977789	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)
10	0.03125000024835268656	0.03125000024835268656	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)

Table 6: Absolute error of NM with [Adeyeye, and Omar. (2019), Kuboye *et al*. (2022), Adoghe and Omole (2019), Ahamad and Charan (2019), Ukpebor *et al*. (2020), Atabo and Adee (2021)] for example 3

t	**ENM**				**ENM**				
	$h=0.103125$	$h=0.003125$	$h=0.01$	$h=0.1$	E [Adeyeye, and Omar. (2019)] $h=0.1$	E[Kuboye *et al.* (2022)] $h=0.1$	E[Ahamad and Charan (2019)] $h=0.1$	E[Ukpebor *et al.* (2020)] $h=0.1$	E[Atabo and Adee (2021)] $h=0.1$
1	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	2.9976(-15)	0.0000(00)	0.0000(00)
2	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	2.7756(-17)	0.0000(00)	2.9976(-15)	0.0000(00)	0.0000(00)
3	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	5.5511(-17)	0.0000(00)	1.8225(-05)	0.0000(00)	0.0000(00)
4	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	5.5511(-17)	7.6800(-05)	0.0000(00)	0.0000(00)
5	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	1.1102(-16)	6.9944(-15)	0.0000(00)	0.0000(00)
6	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	1.1102(-16)	0.0000(00)	0.0000(00)	0.0000(00)
7	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	2.2205(-16)	2.9976(-15)	0.0000(00)	0.0000(00)
8	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	2.9976(-15)	2.0000(-18)	0.0000(00)
9	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	1.1102(-16)	0.0000(00)	2.0000(-18)	1.1100(-16)
10	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	0.0000(00)	2.2205(-16)	7.5000(-03)	1.0000(-17)	0.0000(00)

5 Discussion and Conclusion

The block approach was utilized to develop a new scheme with a generalized eight-point formula. This new scheme was directly applied to various third and fourth-order oscillatory differential equations, including physical application problems, highly stiff, and non-stiff oscillatory differential equations. Comparisons with existing methods such as those by Omar *et al*. (2016), Kayode and Obaruha (2017), Adeyeye and Omar (2019), Fasasi *et al*. (2015), Areo *et al*. (2017), Sunday (2018), Sunday *et al*. (2019), Adoghe and Omole (2019), Adeyeye and Omar (2019), Allogmany *et al*. (2020), Kuboye and Omar (2015), Ahamad and Charan (2019), Ukpebor *et al*. (2020), Atabo and Adee (2021) were conducted by varying the step size.

The results indicated that the new scheme demonstrated better convergence and accuracy compared to these existing methods. The higher-order scheme, derived from the general $k-step$ block approach using the linear block approach, proved to be straightforward to implement. Numerical analysis confirmed that the new scheme is consistent and zero-stable, ensuring its convergence.

Conflict of Interest

The authors affirm that the publishing of this paper does not involve any conflicts of interest

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