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The Efficiency of Block Hybrid Method for Solving Malthusian Growth Model and Prothero-Robinson Oscillatory Differential Equations

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ABSTRACT

The efficiency of block hybrid method for solving Malthusian Growth Model, Prothero-Robinson equation and highly stiff oscillatory differential equations was proposed using a power series polynomial through interpolation and collocation. The new method's basic properties, including order, error constant, consistency, zero-stability, and stability regions, were comprehensively analyzed and satisfied all necessary conditions for analysis. Tested on various real-life problems, the new method demonstrated superior performance compared to existing techniques. The study highlights the innovative approach's enhanced convergence and stability properties, providing a more reliable numerical analysis tool for researchers and practitioners. Practical applications validate the method's effectiveness, showcasing its superior performance across different examples and establishing it as a highly effective solution for Malthusian growth model and oscillatory differential equations.

1. Introduction

Oscillatory differential equations, characterized by periodic or quasi-periodic solutions, are crucial in modeling various natural and engineered systems such as mechanical vibrations, electrical circuits, and biological rhythms (Lambert, 1991). The oscillatory differential equations often take the form of second-order linear differential equations with applications including the simple harmonic oscillator in mechanics, LC circuits in electrical engineering, and predator-prey dynamics in biology (Lambert, 1991; Sabo, 2021; Kwari *et al.,* 2023). Analytical solutions are feasible for simpler cases, while numerical methods like the Runge-Kutta and block methods are employed for more complex or stiff equations. Recent advancements in numerical techniques, particularly for stiff problems, have enhanced stability and efficiency, making these methods indispensable for accurate and reliable solutions in diverse applications (Sabo, 2021).

In line with this approach, the solution to oscillatory for initial value problems (IVPs) of first-order epidemic model differential equations (ODEs) in the following form:

$$u'(v) = f(v, u); u(v_0) = u_0 \tag{1}$$

The Malthusian growth model, named after British economist Thomas Robert Malthus, is a theory of population growth that highlights the potential for exponential population increase and its implications for resources. Malthus first presented his ideas in his 1798 work, "An Essay on the Principle of Population." At its core, the model posits that populations tend to grow exponentially when resources are abundant, meaning that the population size can double at a constant rate over successive time periods, assuming no constraints (Malthus, 1798). This exponential growth of the population starkly contrasts with the arithmetic growth of food supply or resources, which Malthus believed increased by a fixed amount each period rather than multiplying. The Malthusian theory explained that the human population grows more rapidly than the food supply until famines, wars, or diseases reduce the population (Lotka, 1934; Oluwaseun and Zurni, 2022).

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The Malthusian growth model can be described using a simple differential equation:

$$\frac{dP}{dt} = rP \tag{2}$$

where, P is the population size and r is the growth rate.

$\frac{dP}{dt}$ is the rate of change of the population over time.

In contemporary applications, the Malthusian model proves valuable across multiple fields. In ecology, it helps in understanding population dynamics and carrying capacity, which are crucial for wildlife conservation and ecosystem management. Economists use the model to analyze the relationship between population growth and resource consumption, aiding in the development of sustainable policies for resource allocation (Sunday *et al.,* 2014). In agriculture, the model underscores the importance of increasing food production to meet the demands of a growing population and supports sustainable farming practices (Yunusa *et al.,* 2024). Public health professionals apply the model to forecast healthcare needs based on population trends, assisting in resource management during pandemics and planning for healthcare infrastructure (Yunusa *et al.,* 2024). Urban planners utilize the model to predict city growth and infrastructure needs, fostering sustainable urban development (Sunday *et al.,* 2015). Globally, the model contributes to climate change discussions by emphasizing sustainable practices to reduce environmental impact and addressing global inequalities in resource distribution. Modern adaptations of the model incorporate technological advancements, economic factors, and behavioral changes, maintaining its relevance in addressing contemporary challenges in sustainability and resource management (Yunusa *et al.,* 2024).

Block methods are advanced numerical techniques for solving ordinary differential equations (ODEs), particularly effective for stiff oscillatory differential equations. These methods extend the concept of linear multi-step methods by computing solutions at multiple points simultaneously within a block, utilizing interpolation and collocation techniques with polynomial functions (Omeje *et al.,* 2021; Ababneh *et al.,* 2022; Raymond and Sabo, 2023). This approach allows for the derivation of a system of algebraic equations that is solved to obtain the values of the solution at various points within each block, enhancing both stability and efficiency compared to traditional single-step methods like Euler's or Runge-Kutta methods (Sabo, 2021; Aloko *et al.,* 2024).

The primary advantages of block methods include superior stability, especially for stiff ODEs, and improved computational efficiency due to the simultaneous computation of multiple points (Kamoh *et al.,* 2017). This makes block methods particularly suitable for large-scale problems and systems with complex dynamics, where traditional methods might require excessively small time steps for stability. Additionally, block methods can achieve higher-order accuracy through the use of higher-degree polynomials for interpolation, leading to more precise solutions with fewer computational steps (Kamoh *et al.,* 2023; Adewale and Sabo, 2024).

Recent research and developments in block methods have focused on enhancing their adaptability and robustness. Adaptive block methods that adjust the block size based on the solution's behavior have been developed to balance computational efficiency and stability (Abolarin *et al.,* 2020). Hybrid block methods that combine block techniques with other numerical approaches are also being explored to further improve accuracy and robustness. Moreover, parallel implementations of block methods leverage modern high-performance computing architectures to accelerate the solution process for large-scale problems, making them an indispensable tool in fields like computational biology, fluid dynamics, and electrical circuit simulation (Kamoh *et al.,* 2017; Sabo *et al.,* 2020; Abolarin *et al.,* 2020; Adewale and Sabo, 2024, Donald *et al.,* 2024). In this study, we will develop a two-step block hybrid method with six of-grid points using power series polynomial as a basic function through interpolation and collocation method.

2 Methodology

The power series as an approximate solution of the form;

$$y(x) = h\sum_{i=0}^{m+n-1} \alpha_j \chi^i \tag{3}$$

is consider as an approximate solution of (1), where m and n are distinct point of interpolation and collocation. Differentiate (3) once to yield,

$$\frac{dy}{dx} = h\sum_{i=0}^{m+n-1} i\alpha_j \chi^{i-1} \tag{4}$$

where $\alpha \in \Re$ for $i = 0\left(\frac{1}{4}\right)2$ and $y(x)$ is continuously differential. Let the solution of (1) be sought on the integration interval $[a,b]$ with a constant step-size h defined by $h = \chi_{n+1} - \chi_n$, $n = 0, 1, \cdots, N$.

Substituting equation (4) into (1) gives,

$$f(x, y) = h\sum_{i=0}^{m+n-1} i\alpha_j \chi^{i-1} \tag{5}$$

We interpolate equation (4) at point, x_{n+m}, $m = \frac{1}{4}$ and collocate equation (5) at points

x_{n+n}, $n = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$ to give,

$$A\chi = U \tag{6}$$

where

$$A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9]^T, U = \left[y_{n+\frac{1}{4}}, f_n, f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+1}, f_{n+\frac{5}{4}}, f_{n+\frac{3}{2}}, f_{n+\frac{7}{4}}, f_{n+2} \right]^T$$

$$X = \begin{bmatrix}
\chi^0_{n+\frac{1}{4}} & \chi^1_{n+\frac{1}{4}} & \chi^2_{n+\frac{1}{4}} & \chi^3_{n+\frac{1}{4}} & \chi^4_{n+\frac{1}{4}} & \chi^5_{n+\frac{1}{4}} & \chi^6_{n+\frac{1}{4}} & \chi^7_{n+\frac{1}{4}} & \chi^8_{n+\frac{1}{4}} & \chi^9_{n+\frac{1}{4}} \\
0 & 1\chi^0_n & 2\chi^1_n & 3\chi^2_n & 4\chi^3_n & 5\chi^4_n & 6\chi^5_n & 7\chi^6_n & 8\chi^7_n & 9\chi^8_n \\
0 & 1\chi^0_{n+\frac{1}{4}} & 2\chi^1_{n+\frac{1}{4}} & 3\chi^2_{n+\frac{1}{4}} & 4\chi^3_{n+\frac{1}{4}} & 5\chi^4_{n+\frac{1}{4}} & 6\chi^5_{n+\frac{1}{4}} & 7\chi^6_{n+\frac{1}{4}} & 8\chi^7_{n+\frac{1}{4}} & 9\chi^8_{n+\frac{1}{4}} \\
0 & 1\chi^0_{n+\frac{1}{2}} & 2\chi^1_{n+\frac{1}{2}} & 3\chi^2_{n+\frac{1}{2}} & 4\chi^3_{n+\frac{1}{2}} & 5\chi^4_{n+\frac{1}{2}} & 6\chi^5_{n+\frac{1}{2}} & 7\chi^6_{n+\frac{1}{2}} & 8\chi^7_{n+\frac{1}{2}} & 9\chi^8_{n+\frac{1}{2}} \\
0 & 1\chi^0_{n+1} & 2\chi^1_{n+1} & 3\chi^2_{n+1} & 4\chi^3_{n+1} & 5\chi^4_{n+1} & 6\chi^5_{n+1} & 7\chi^6_{n+1} & 8\chi^7_{n+1} & 9\chi^8_{n+1} \\
0 & 1\chi^0_{n+\frac{3}{4}} & 2\chi^1_{n+\frac{3}{4}} & 3\chi^2_{n+\frac{3}{4}} & 4\chi^3_{n+\frac{3}{4}} & 5\chi^4_{n+\frac{3}{4}} & 6\chi^5_{n+\frac{3}{4}} & 7\chi^6_{n+\frac{3}{4}} & 8\chi^7_{n+\frac{3}{4}} & 9\chi^8_{n+\frac{3}{4}} \\
0 & 1\chi^0_{n+\frac{5}{4}} & 2\chi^1_{n+\frac{5}{4}} & 3\chi^2_{n+\frac{5}{4}} & 4\chi^3_{n+\frac{5}{4}} & 5\chi^4_{n+\frac{5}{4}} & 6\chi^5_{n+\frac{5}{4}} & 7\chi^6_{n+\frac{5}{4}} & 8\chi^7_{n+\frac{5}{4}} & 9\chi^8_{n+\frac{5}{4}} \\
0 & 1\chi^0_{n+\frac{3}{2}} & 2\chi^1_{n+\frac{3}{2}} & 3\chi^2_{n+\frac{3}{2}} & 4\chi^3_{n+\frac{3}{2}} & 5\chi^4_{n+\frac{3}{2}} & 6\chi^5_{n+\frac{3}{2}} & 7\chi^6_{n+\frac{3}{2}} & 8\chi^7_{n+\frac{3}{2}} & 9\chi^8_{n+\frac{3}{2}} \\
0 & 1\chi^0_{n+\frac{7}{4}} & 2\chi^1_{n+\frac{7}{4}} & 3\chi^2_{n+\frac{7}{4}} & 4\chi^3_{n+\frac{7}{4}} & 5\chi^4_{n+\frac{7}{4}} & 6\chi^5_{n+\frac{7}{4}} & 7\chi^6_{n+\frac{7}{4}} & 8\chi^7_{n+\frac{7}{4}} & 9\chi^8_{n+\frac{7}{4}} \\
0 & 1\chi^0_{n+2} & 2\chi^1_{n+2} & 3\chi^2_{n+2} & 4\chi^3_{n+2} & 5\chi^4_{n+2} & 6\chi^5_{n+2} & 7\chi^6_{n+2} & 8\chi^7_{n+2} & 9\chi^8_{n+2}
\end{bmatrix}$$

Solving (6), for α_i, $i = 0\left(\frac{1}{4}\right)2$ and replacing back into (3) gives a linear block scheme as

$$y(t) = \alpha_{\frac{1}{4}}(t) y_{n+\frac{1}{4}} + h\begin{bmatrix} \beta_0(t) f_n + \beta_{\frac{1}{4}}(t) f_{n+\frac{1}{4}} + \beta_{\frac{1}{2}}(t) f_{n+\frac{1}{2}} + \beta_{\frac{3}{4}}(t) f_{n+\frac{3}{4}} \\ + \beta_1(t) f_{n+1} + \beta_{\frac{5}{4}}(t) f_{n+\frac{5}{4}} + \beta_{\frac{3}{2}}(t) f_{n+\frac{3}{2}} + \beta_{\frac{7}{4}}(t) f_{n+\frac{7}{4}} + \beta_2(t) f_{n+2} \end{bmatrix} \tag{7}$$

where

$$\alpha_{\frac{1}{4}}=1$$

$$\beta_0=\frac{1}{14515200}\begin{pmatrix}-1070017+5955064t-14139696t^2+18399488t^3-14124800t^4+\\6389760t^5-1576960t^6+163840t^7\end{pmatrix}(4t-1)^2$$

$$\beta_{\frac{1}{4}}=\frac{1}{7257600}\begin{pmatrix}-2233547-8934188t+80384848t^2-210408128t^3+284101888t^4-\\221957120t^5+101355520t^6-25149440t^7+2621440t^8\end{pmatrix}(4t-1)$$

$$\beta_{\frac{1}{2}}=\frac{1}{7257600}\begin{pmatrix}2302297+18418376t-92702544t^2+165544192t^3-152377600t^4+\\77568000t^5-20787200t^6+2293760t^7\end{pmatrix}(4t-1)^2$$

$$\beta_{\frac{3}{4}}=\frac{1}{7257600}\begin{pmatrix}2797679+22381432t-136661808t^2+271503104t^3-268832000t^4+\\144353280t^5-40284160t^6+4587520t^7\end{pmatrix}(4t-1)^2$$

$$\beta_1=\frac{1}{90720}\begin{pmatrix}31457+22381432t-136661808t^2+271503104t^3-268832000t^4+\\144353280t^5-40284160t^6+4587520t^7\end{pmatrix}(4t-1)^2$$

$$\beta_{\frac{5}{4}}=\frac{1}{7257600}\begin{pmatrix}251656+1665264t-136661808t^2+271503104t^3-268832000t^4+\\144353280t^5-40284160t^6+4587520t^7\end{pmatrix}(4t-1)^2$$

$$\beta_{\frac{3}{2}}=\frac{1}{7257600}\begin{pmatrix}645607+5164856t-36748464t^2+84205312t^3-95553280t^4+\\58214400t^5-18206720t^6+2293760t^7\end{pmatrix}(4t-1)^2$$

$$\beta_{\frac{6}{4}}=\frac{1}{7257600}\begin{pmatrix}156437+1251496t-9079824t^2+21247232t^3-24723200t^4+\\15513600t^5-5017600t^6+655360t^7\end{pmatrix}(4t-1)^2$$

$$\beta_2=\frac{1}{14515200}\begin{pmatrix}33953+271624t-1999056t^2+4752128t^3-5638400t^4+\\3624960t^5-1208320t^6+163840t^7\end{pmatrix}(4t-1)^2$$

for $t=\frac{(\chi-\chi_n)}{h}$.

Evaluating (7) at non-interpolating points to gives

$$\left.\begin{aligned}
y_n&=y_{n+\frac{1}{4}}+\psi_{101}f_n+\psi_{102}f_{n+\frac{1}{4}}+\psi_{103}f_{n+\frac{1}{2}}+\psi_{104}f_{n+\frac{3}{4}}+\psi_{105}f_{n+1}+\psi_{106}f_{n+\frac{5}{4}}+\psi_{107}f_{n+\frac{3}{2}}+\psi_{108}f_{n+\frac{7}{4}}+\psi_{109}f_{n+2}\\
y_{n+\frac{1}{2}}&=y_{n+\frac{1}{4}}+\psi_{111}f_n+\psi_{112}f_{n+\frac{1}{4}}+\psi_{113}f_{n+\frac{1}{2}}+\psi_{114}f_{n+\frac{3}{4}}+\psi_{115}f_{n+1}+\psi_{116}f_{n+\frac{5}{4}}+\psi_{117}f_{n+\frac{3}{2}}+\psi_{118}f_{n+\frac{7}{4}}+\psi_{119}f_{n+2}\\
y_{n+\frac{3}{4}}&=y_{n+\frac{1}{4}}+\psi_{121}f_n+\psi_{122}f_{n+\frac{1}{4}}+\psi_{123}f_{n+\frac{1}{2}}+\psi_{124}f_{n+\frac{3}{4}}+\psi_{125}f_{n+1}+\psi_{126}f_{n+\frac{5}{4}}+\psi_{127}f_{n+\frac{3}{2}}+\psi_{128}f_{n+\frac{7}{4}}+\psi_{129}f_{n+2}\\
y_{n+1}&=y_{n+\frac{1}{4}}+\psi_{131}f_n+\psi_{132}f_{n+\frac{1}{4}}+\psi_{133}f_{n+\frac{1}{2}}+\psi_{134}f_{n+\frac{3}{4}}+\psi_{135}f_{n+1}+\psi_{136}f_{n+\frac{5}{4}}+\psi_{137}f_{n+\frac{3}{2}}+\psi_{138}f_{n+\frac{7}{4}}+\psi_{139}f_{n+2}\\
y_{n+\frac{5}{4}}&=y_{n+\frac{1}{4}}+\psi_{141}f_n+\psi_{142}f_{n+\frac{1}{4}}+\psi_{143}f_{n+\frac{1}{2}}+\psi_{144}f_{n+\frac{3}{4}}+\psi_{145}f_{n+1}+\psi_{146}f_{n+\frac{5}{4}}+\psi_{147}f_{n+\frac{3}{2}}+\psi_{148}f_{n+\frac{7}{4}}+\psi_{149}f_{n+2}\\
y_{n+\frac{3}{2}}&=y_{n+\frac{1}{4}}+\psi_{151}f_n+\psi_{152}f_{n+\frac{1}{4}}+\psi_{153}f_{n+\frac{1}{2}}+\psi_{154}f_{n+\frac{3}{4}}+\psi_{155}f_{n+1}+\psi_{156}f_{n+\frac{5}{4}}+\psi_{157}f_{n+\frac{3}{2}}+\psi_{158}f_{n+\frac{7}{4}}+\psi_{159}f_{n+2}\\
y_{n+\frac{7}{4}}&=y_{n+\frac{1}{4}}+\psi_{161}f_n+\psi_{162}f_{n+\frac{1}{4}}+\psi_{163}f_{n+\frac{1}{2}}+\psi_{164}f_{n+\frac{3}{4}}+\psi_{165}f_{n+1}+\psi_{166}f_{n+\frac{5}{4}}+\psi_{167}f_{n+\frac{3}{2}}+\psi_{168}f_{n+\frac{7}{4}}+\psi_{169}f_{n+2}\\
y_{n+1}&=y_{n+\frac{1}{4}}+\psi_{171}f_n+\psi_{172}f_{n+\frac{1}{4}}+\psi_{173}f_{n+\frac{1}{2}}+\psi_{174}f_{n+\frac{3}{4}}+\psi_{175}f_{n+1}+\psi_{176}f_{n+\frac{5}{4}}+\psi_{177}f_{n+\frac{3}{2}}+\psi_{178}f_{n+\frac{7}{4}}+\psi_{179}f_{n+2}
\end{aligned}\right\}\quad(8)$$

Evaluating (7) at $t=\left[\frac{1}{4},\frac{1}{2},\frac{3}{4},1,\frac{5}{4},\frac{3}{2},\frac{7}{4},2\right]$ points to gives discrete block scheme of the form:

$$
\left.\begin{aligned}
y_{n+\frac{1}{4}} &= y_n + \Omega_{101} f_n + \Omega_{102} f_{n+\frac{1}{4}} + \Omega_{103} f_{n+\frac{1}{2}} + \Omega_{104} f_{n+\frac{3}{4}} + \Omega_{105} f_{n+1} + \Omega_{106} f_{n+\frac{5}{4}} + \Omega_{107} f_{n+\frac{3}{2}} + \Omega_{108} f_{n+\frac{7}{4}} + \Omega_{109} f_{n+2} \\
y_{n+\frac{1}{2}} &= y_n + \Omega_{111} f_n + \Omega_{112} f_{n+\frac{1}{4}} + \Omega_{113} f_{n+\frac{1}{2}} + \Omega_{114} f_{n+\frac{3}{4}} + \Omega_{115} f_{n+1} + \Omega_{116} f_{n+\frac{5}{4}} + \Omega_{117} f_{n+\frac{3}{2}} + \Omega_{118} f_{n+\frac{7}{4}} + \Omega_{119} f_{n+2} \\
y_{n+\frac{3}{4}} &= y_n + \Omega_{121} f_n + \Omega_{122} f_{n+\frac{1}{4}} + \Omega_{123} f_{n+\frac{1}{2}} + \Omega_{124} f_{n+\frac{3}{4}} + \Omega_{125} f_{n+1} + \Omega_{126} f_{n+\frac{5}{4}} + \Omega_{127} f_{n+\frac{3}{2}} + \Omega_{128} f_{n+\frac{7}{4}} + \Omega_{129} f_{n+2} \\
y_{n+1} &= y_n + \Omega_{131} f_n + \Omega_{132} f_{n+\frac{1}{4}} + \Omega_{133} f_{n+\frac{1}{2}} + \Omega_{134} f_{n+\frac{3}{4}} + \Omega_{135} f_{n+1} + \Omega_{136} f_{n+\frac{5}{4}} + \Omega_{137} f_{n+\frac{3}{2}} + \Omega_{138} f_{n+\frac{7}{4}} + \Omega_{139} f_{n+2} \\
y_{n+\frac{5}{4}} &= y_n + \Omega_{141} f_n + \Omega_{142} f_{n+\frac{1}{4}} + \Omega_{143} f_{n+\frac{1}{2}} + \Omega_{144} f_{n+\frac{3}{4}} + \Omega_{145} f_{n+1} + \Omega_{146} f_{n+\frac{5}{4}} + \Omega_{147} f_{n+\frac{3}{2}} + \Omega_{148} f_{n+\frac{7}{4}} + \Omega_{149} f_{n+2} \\
y_{n+\frac{3}{2}} &= y_n + \Omega_{151} f_n + \Omega_{152} f_{n+\frac{1}{4}} + \Omega_{153} f_{n+\frac{1}{2}} + \Omega_{154} f_{n+\frac{3}{4}} + \Omega_{155} f_{n+1} + \Omega_{156} f_{n+\frac{5}{4}} + \Omega_{157} f_{n+\frac{3}{2}} + \Omega_{158} f_{n+\frac{7}{4}} + \Omega_{159} f_{n+2} \\
y_{n+\frac{7}{4}} &= y_n + \Omega_{161} f_n + \Omega_{162} f_{n+\frac{1}{4}} + \Omega_{163} f_{n+\frac{1}{2}} + \Omega_{164} f_{n+\frac{3}{4}} + \Omega_{165} f_{n+1} + \Omega_{166} f_{n+\frac{5}{4}} + \Omega_{167} f_{n+\frac{3}{2}} + \Omega_{168} f_{n+\frac{7}{4}} + \Omega_{169} f_{n+2} \\
y_{n+1} &= y_n + \Omega_{171} f_n + \Omega_{172} f_{n+\frac{1}{4}} + \Omega_{173} f_{n+\frac{1}{2}} + \Omega_{174} f_{n+\frac{3}{4}} + \Omega_{175} f_{n+1} + \Omega_{176} f_{n+\frac{5}{4}} + \Omega_{177} f_{n+\frac{3}{2}} + \Omega_{178} f_{n+\frac{7}{4}} + \Omega_{179} f_{n+2}
\end{aligned}\right\} \quad (9)
$$

3 Analysis of the new method

We will examine the basic properties of the new method in this section.

3.1 Order and Error Constant

This subsection establishes the linear operator $\ell[y(x_i);h]$ associated with the newly derived method (Donald *et al.,* 2024).

Definition 1

A block hybrid method is of order p if it satisfies the condition

$$c_0 = c_1 = c_2 = c_3 = \cdots = c_p = c_{p+1} = 0,\ c_{p+2} \neq 0,$$

where

$$
\left.\begin{aligned}
c_0 &= \sum_{j=0}^{k} \alpha_j \\
c_1 &= \sum_{j=0}^{k} \left(j\alpha_j - \beta_j \right) \\
&. \\
&. \\
&. \\
c_p &= \sum_{j=0}^{k} \left[\frac{1}{p!} j^p \alpha_j - \frac{1}{(p-1)!} \left(j^{p-1} \beta_j \right) \right],\ p = 2, 3, \cdots, q+1
\end{aligned}\right\} \quad (10)
$$

the parameter $c_{p+2} \neq 0$ is referred to as the error constant with the local truncation error defined as

$$\begin{bmatrix}
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{2233547}{7257600}\left(\frac{1}{4}\right)-\frac{2302297}{7257600}\left(\frac{1}{2}\right)-\frac{2797679h}{7257600}\left(\frac{3}{4}\right)-\frac{31457h}{90720}(1)-\frac{1573169h}{7257600}\left(\frac{5}{4}\right)-\frac{645607h}{7257600}\left(\frac{3}{2}\right)-\frac{156437h}{7257600}\left(\frac{7}{4}\right)-\frac{33953h}{14515200}(2)\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{22823}{56700}\left(\frac{1}{4}\right)+\frac{21247}{226800}\left(\frac{1}{2}\right)-\frac{15011h}{56700}\left(\frac{3}{4}\right)+\frac{2903h}{11340}(1)-\frac{9341h}{56700}\left(\frac{5}{4}\right)+\frac{15577h}{226800}\left(\frac{3}{2}\right)-\frac{953h}{56700}\left(\frac{7}{4}\right)+\frac{119h}{64800}(2)\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{3}{4}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{35451}{89600}\left(\frac{1}{4}\right)-\frac{1719h}{89600}\left(\frac{1}{2}\right)-\frac{39967h}{89600}\left(\frac{3}{4}\right)+\frac{351h}{1120}(1)-\frac{17217h}{89600}\left(\frac{5}{4}\right)+\frac{7031h}{89600}\left(\frac{3}{2}\right)-\frac{243h}{12800}\left(\frac{7}{4}\right)+\frac{369h}{179200}(2)\right]\\
\sum_{j=0}^{\infty}\frac{(1)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{5644}{14175}\left(\frac{1}{4}\right)-\frac{61h}{14175}\left(\frac{1}{2}\right)-\frac{8188h}{14175}\left(\frac{3}{4}\right)+\frac{454h}{2835}(1)-\frac{2308h}{14175}\left(\frac{5}{4}\right)+\frac{989h}{14175}\left(\frac{3}{2}\right)-\frac{244h}{14175}\left(\frac{7}{4}\right)+\frac{107h}{56700}(2)\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{5}{4}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{115075}{290304}\left(\frac{1}{4}\right)-\frac{3775h}{290304}\left(\frac{1}{2}\right)-\frac{159175h}{290304}\left(\frac{3}{4}\right)+\frac{125h}{18144}(1)-\frac{85465h}{290304}\left(\frac{5}{4}\right)+\frac{24575h}{290304}\left(\frac{3}{2}\right)-\frac{5725h}{290304}\left(\frac{7}{4}\right)+\frac{175h}{82944}(2)\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{3}{2}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{279}{700}\left(\frac{1}{4}\right)-\frac{9h}{2800}\left(\frac{1}{2}\right)-\frac{403h}{700}\left(\frac{3}{4}\right)-\frac{9h}{140}(1)-\frac{333h}{700}\left(\frac{5}{4}\right)+\frac{79h}{2800}\left(\frac{3}{2}\right)-\frac{9h}{700}\left(\frac{7}{4}\right)+\frac{9h}{5600}(2)\right]\\
\sum_{j=0}^{\infty}\frac{\left(\frac{7}{4}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{408317}{1036800}\left(\frac{1}{4}\right)-\frac{24353h}{1036800}\left(\frac{1}{2}\right)-\frac{542969h}{1036800}\left(\frac{3}{4}\right)-\frac{343h}{12960}(1)-\frac{368039h}{1036800}\left(\frac{5}{4}\right)+\frac{261023h}{1036800}\left(\frac{3}{2}\right)-\frac{111587h}{1036800}\left(\frac{7}{4}\right)+\frac{8183h}{2073600}(2)\right]\\
\sum_{j=0}^{\infty}\frac{(2)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{5888}{14175}\left(\frac{1}{4}\right)-\frac{928h}{14175}\left(\frac{1}{2}\right)-\frac{10496h}{14175}\left(\frac{3}{4}\right)-\frac{908h}{2835}(1)-\frac{10496h}{14175}\left(\frac{5}{4}\right)-\frac{928h}{14175}\left(\frac{3}{2}\right)-\frac{5888h}{14175}\left(\frac{7}{4}\right)-\frac{989h}{14175}(2)\right]
\end{bmatrix}$$

Therefore, the new method is of uniform order ten as well as error constant is given by

$$C_{11}=\begin{pmatrix}7.5269\times10^{-09}\\6.1308\times10^{-09}\\6.6523\times10^{-09}\\6.3242\times10^{-09}\\6.6523\times10^{-09}\\6.1308\times10^{-09}\\7.5269\times10^{-06}\\6.6523\times10^{-09}\end{pmatrix}$$

3.2 Consistent

Theorem 1:

The block hybrid method is said to be consistent if its order p is greater than or equal to one $(p \geq 1)$. Therefore, the new method is consistent because the order of the method is order greater than one (Adewale and Sabo, 2024).

3.3 Zero Stable

By definition, the new method is said to be zero stable as $h \to 0$ if the roots of the polynomial $\pi(r)=0$ satisfy $\left|\left[\sum A^0 R^{k-1}\right]\right| \leq 1,$ and those roots with R = 1 must be simple. Hence it's found as

$$\pi(r)=\left| r\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{bmatrix}\right|=\begin{vmatrix}-r&0&0&0&0&0&0&1\\0&-r&0&0&0&0&0&1\\0&0&-r&0&0&0&0&1\\0&0&0&-r&0&0&0&1\\0&0&0&0&-r&0&0&1\\0&0&0&0&0&-r&0&1\\0&0&0&0&0&0&-r&1\\0&0&0&0&0&0&0&1-r\end{vmatrix}=r^8(r-1)$$

Then, solving for r in $r^8(r-1)$,gives $r=0, 0,0, 0,0,0, 0,1$. Therefore, the method is zero stable.

3.4 Convergence

Theorem 2

Dahlquist's theorem states that, the block hybrid method is said to be convergent if it is convergent consistent and zero-stability (Lambert, 1991).

Therefore, by theorem 2 the new method is convergent.

3.5 Region of Absolute Stability

The boundary locus method is used to generate the new method stability polynomial (Adewale and Sabo, 2024). The polynomial is defined as

$$\begin{aligned}\bar{h}(w)=&\,h^{18}\left(\frac{1}{2153335465\,642}w^5-\frac{6756}{4492065676\,6700}w^4\right)-h^{16}\left(\frac{2}{8667898778\,76}w^5-\frac{998}{6754554523\,33}w^4\right)\\&+h^{14}\left(\frac{1}{99809}w^4-\frac{2342}{1232120000}w^5\right)+h^{12}\left(\frac{111}{43333280}w^4-\frac{1}{9087123333}w^5\right)+h^{10}\left(\frac{23}{95435}w^4-\frac{39}{2211130}w^5\right)\\&+h^{8}\left(\frac{1}{4445553}w^4-\frac{12}{2220101}w^5\right)+h^{6}\left(\frac{9}{30000000}w^4-\frac{100}{12187095}w^5\right)+h^{4}\left(\frac{1}{4455786}w^4-\frac{9987}{34532464}w^5\right)\\&-\frac{11}{610}h^2w^4+w^5-\frac{1}{21}w^3\end{aligned} \quad (11)$$

The region of absolute stability of new method is a region in the complex z plane. The numerical solution of new method satisfies $y_j \to 0$ *as* $j \to \infty$ for any initial condition. The stability region obtained in Figure 1 is $A-stable$. The polynomial is used to plot the region as

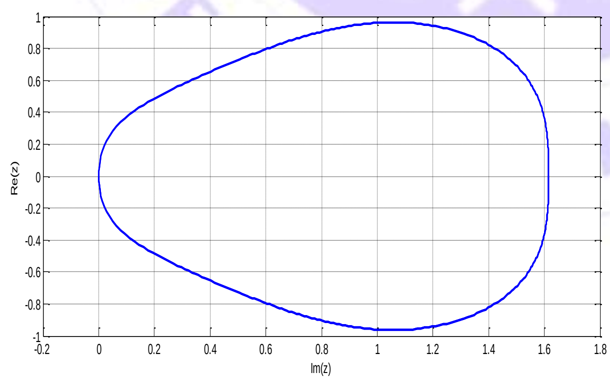

Figure 1: Showing the region of stability.

4. Numerical Examples

In order to study the efficiency of the developed methods, the performance of the new method is tested on five sampled problems (specifically, Malthusian Growth Model, Prothero-Robinson oscillatory differential equation and some highly stiff oscillatory differential equations). For each problem, the approximate solutions are compared with numerical results and the absolute error obtained from numerical application of new method will be compared with existing methods of Sunday *et al.,* (2013), Okunuga *et al.,* (2013), Sunday *et al.,* (2014), Sunday *et al.,* (2015), Omar and Adeyeye (2016), Skwame *et al.,* (2018), Oluwaseun and Zurni, (2022) and Yunusa *et al.,* (2024).

The following acronyms will be used in the tables:

ENM: Means Error in New Method
ESe13: Means Error in Sunday *et al.,* (2013)
EOe13: Means Error in Okunuga *et al.,* (2013)
ESe14: Means Error in Sunday *et al.,* (2014)
ESe15: Means Error in Sunday *et al.,* (2015)
EOA16: Means Error in Omar and Adeyeye (2016)
ESe18: Means Error in Skwame *et al.,* (2018).
EOZ22: Means Error in Oluwaseun, and Zurni, (2022)
EYe24: Means Error in Yunusa *et al.,* (2024)

Example 1: (Malthus Growth Model)

The Malthusian growth model can be described using a simple differential equation:

$$\frac{du}{dv} = kp,\ v \in [0,1], \tag{12}$$

with the exact solution given by

$$u(v) = 100\exp(0.250679566129x) \tag{13}$$

Initial condition $u(0) = 100$ with $k = 0.250679566129$ and $h = 0.1$

Source: [Oluwaseun, and Zurni, (2022); Yunusa *et al.,* (2024)].

Example 2: **(Prothero-Robinson Equation)**

Let consider the Prothero-Robinson oscillatory differential equation, which was addressed by Sunday *et al.,* (2014) and Sunday *et al.,* (2015), formulated as follows:

$$y' = \Phi(y - \sin x) - y,\ \Phi = -1,\ y(0) = 0 \tag{14}$$

Which has the exact solution as

$$y(x) = \sin x \tag{15}$$

Example 3: Consider the highly stiff oscillatory differential equation

$$\frac{du}{dv} = -\sin(v) - 200(u - \cos(v)),\ h = 0.01,\ u(0) = 0 \tag{16}$$

with the exact solution

$$u(v) = \cos(v) - e^{-200v} \tag{17}$$

Source: [Sunday *et al.,* (2013), Sunday *et al.,* (2015)]

Example 4: Consider the highly stiff oscillatory differential equation

$$\frac{du}{dv} = -10(u-1)^2,\ h = 0.01,\ u(0) = 2 \tag{18}$$

with the exact solution

$$u(v) = 1 + \frac{1}{1+10v} \tag{19}$$

Source: [Sunday *et al.,* (2013); Okunuga *et al.,* (2013)].

Example 5: Consider the highly stiff oscillatory differential equation

$$\frac{du}{dv} = -\psi u,\ h = 0.1,\ u(0) = \psi = 1 \tag{20}$$

with the exact solution

$$u(v) = \exp(-v) \tag{21}$$

Source: [Omar and Adeyeye, (2016); Skwame *et al.,* (2018)]

Table 1: The results of example 1 with Oluwaseun, and Zurni, (2022); Yunusa *et al.,* (2024).

V	**Exact Solution**	**Computed Solution**	**ENM**	**EOZ22**	**EYe24**
0.1	102.53847998347329794000	102.53847998347329795000	1.0000(-17)	1.6677(-08)	0.0000(00)
0.2	105.14139877321154182000	105.14139877321154182000	0.0000(00)	4.4003(-10)	0.0000(00)
0.3	107.81039213541335645000	107.81039213541335645000	0.0000(00)	1.7117(-08)	0.0000(00)
0.4	110.54713735987489512000	110.54713735987489512000	0.0000(00)	8.8005(-10)	0.0000(00)
0.5	113.35335431405805132000	113.35335431405805134000	2.0000(-17)	1.7557(-08)	1.4211(-14)
0.6	116.23080652391598100000	116.23080652391598100000	0.0000(00)	1.3201(-09)	1.4211(-14)
0.7	119.18130228215516429000	119.18130228215516432000	3.0000(-17)	1.7997(-08)	1.4211(-14)
0.8	122.20669578463047796000	122.20669578463047798000	2.0000(-17)	1.7601(-09)	0.0000(00)
0.9	125.30888829558742918000	125.30888829558742922000	4.0000(-17)	1.8437(-08)	1.4211(-14)
1.0	128.48982934248383035000	128.48982934248383035000	0.0000(00)	2.2001(-09)	0.0000(00)

Table 2: The results of example 2 with Sunday *et al.,* (2014), Sunday *et al.,* (2015)

V	**Exact Solution**	**Computed Solution**	**ENM**	**ESe14**	**ESe15**
0.1	0.09983341664682815231	0.09983341664682815235	4.0000(-20)	1.4530(-11)	1.3422(-11)
0.2	0.19866933079506121546	0.19866933079506121544	2.0000(-20)	1.6211(-11)	2.1464(-11)
0.3	0.29552020666133957511	0.29552020666133957522	1.1000(-19)	2.1310(-11)	3.2359(-11)
0.4	0.38941834230865049167	0.38941834230865049161	6.0000(-20)	1.3799(-11)	4.1877(-11)
0.5	0.47942553860420300027	0.47942553860420300045	1.8000(-19)	2.7441(-11)	4.6377(-11)
0.6	0.56464247339503535720	0.56464247339503535708	1.2000(-19)	1.1114(-11)	5.3368(-11)
0.7	0.64421768723769105367	0.64421768723769105389	2.2000(-19)	2.8657(-11)	5.8936(-11)
0.8	0.71735609089952276163	0.71735609089952276142	2.1000(-19)	1.9218(-10)	6.0221(-11)
0.9	0.78332690962748338846	0.78332690962748338869	2.3000(-19)	1.2392(-10)	6.3342(-11)
1.0	0.84147098480789650665	0.84147098480789650634	3.1000(-19)	1.4711(-10)	6.5059(-11)

Table 3: The results of example 3 with Sunday *et al.,* (2013), Sunday *et al.,* (2015).

V	**Exact Solution**	**Computed Solution**	**ENM**	**ESe13**	**ESe15**
0.001	0.18126874692477177712	0.18126874692206024632	2.7115(-12)	3.7249(-10)	6📫5812(-06)
0.002	0.32967795396412439246	0.32967795396502719904	9.0281(-13)	5.2169(-10)	2.9379(-06)
0.003	0.45118386391042716158	0.45118386390934872361	1.0784(-12)	6.7870(-10)	9.3961(-05)
0.004	0.55066303589223450724	0.55066303589344484589	1.2103(-12)	7.6010(-10)	1.1305(-05)
0.005	0.63210805885482676508	0.63210805885459933723	2.2743(-13)	7.4126(-10)	7.9107(-06)
0.006	0.69878778814058064233	0.69878778814179761365	1.2169(-12)	7.4495(-10)	1.0313(-05)
0.007	0.75337853615825529977	0.75337853615843497422	1.7967(-13)	7.2211(-10)	1.0426(-05)
0.008	0.79807148217492301264	0.79807148217601069296	1.0877(-12)	6.5649(-10)	7.7981(-05)
0.009	0.83466061205144457875	0.83466061205178764743	3.4307(-13)	6.1326(-10)	8.4900(-05)
0.01	0.86461471717914105002	0.86461471718005241732	9.1137(-13)	5📫6367(-	8📫0388(-

	10)	05)

Table 4: The results of example 4 with Sunday *et al.,* (2013); Okunuga *et al.,* (2013).

V	Exact Solution	Computed Solution	ENM	ESe13	EOe13
0.001	1.90909090884750640830	1.90909090909016653110	2.4266(-11)	2.4025(-08)	1.0700(-05)
0.002	1.83333333337241953740	1.83333333333346491070	3.8955(-11)	3.1560(-08)	2.3800(-05)
0.003	1.76923076920944483900	1.76923076923076190480	2.1317(-11)	3.2631(-08)	4.5100(-05)
0.004	1.71428571432193859870	1.71428571428582904000	3.6110(-11)	3.1192(-08)	6.2000(-04)
0.005	1.66666666668304290430	1.66666666666674168450	1.6301(-11)	2.8877(-08)	8.8400(-04)
0.006	1.62500000002955801560	1.62500000000009115730	2.9467(-11)	2.6370(-08)	1.0300(-03)
0.007	1.58823529413888054590	1.58823529411772147080	2.1159(-11)	2.3953(-08)	1.2700(-03)
0.008	1.55555555557943834040	1.55555555555562832480	2.3810(-11)	2.1734(-08)	1.5300(-03)
0.009	1.52631578949329163390	1.52631578947374764750	1.9544(-11)	1.9740(-08)	1.7500(-03)
0.010	1.50000000001952055900	1.50000000000005914080	1.9461(-11)	1.7969(-08)	1.8100(-03)

Table 5: The results of example 5 with Omar and Adeyeye (2016); Skwame, *et al.,* (2018).

V	Exact Solution	Computed Solution	ENM	EOA16	ESe18
0.1	0.90483741803595957316	0.90483741803595957263	5.3000(-19)	9.0730(-12)	5.0000(-10)
0.2	0.81873075307798185867	0.81873075307798185873	6.0000(-20)	1.1768(-11)	5.0000(-10)
0.3	0.74081822068171786607	0.74081822068171786568	3.9000(-19)	2.3144(-11)	8.0000(-10)
0.4	0.67032004603563930074	0.67032004603563930084	1.0000(-19)	2.8440(-11)	7.0000(-10)
0.5	0.60653065971263342360	0.60653065971263342333	2.7000(-19)	3.1815(-11)	1.1000(-09)
0.6	0.54881163609402643263	0.54881163609402643275	1.2000(-19)	3.4927(-11)	1.1000(-09)
0.7	0.49658530379140951470	0.49658530379140951452	1.8000(-19)	3.6582e-11	1.1000(-09)
0.8	0.44932896411722159143	0.44932896411722159156	1.3000(-19)	3.8127(-11)	1.0000e-09
0.9	0.40656965974059911188	0.40656965974059911176	1.2000(-19)	3.8576(-11)	1.0000e-09
1.0	0.36787944117144232160	0.36787944117144232173	1.3000(-19)	3.9020(-11)	1.0000(-09)

4.3 Discussion of Results and Conclusion

The new block hybrid method for solving first-order stiff initial value problems of the form (1) was developed using a power series polynomial through interpolation and collocation. The analysis of the basic properties of the new method, including order, error constant, consistency, zero-stability, and stability regions, has been conducted and all conditions have been satisfied. The newly constructed methods were applied to various real-life problems. The results, displayed in Tables 1 to 5, demonstrate that the new method outperforms the methods proposed by Sunday *et al.,* (2013), Okunuga *et al.,* (2013), Sunday *et al.,* (2014), Sunday *et al.,* (2015), Omar and Adeyeye (2016), Skwame *et al.,* (2018), Oluwaseun and Zurni, (2022) and Yunusa *et al.,* (2024).

Specifically:

i. Example 1 (Malthus growth model): Results in Table 1 show that new method is superior to the methods of Oluwaseun and Zurni (2022) and Yunusa *et al.,* (2024).
ii. Example 2 (Prothero differential equation): Result in Table 2 indicate the better convergence of new method when compared to the methods of Sunday *et al.,* (2014), Sunday *et al.,* (2015).
iii. Example 3 (Highly stiff oscillatory differential equation): as shown in table 3, the new method demonstrates faster convergence than the methods of Sunday *et al.,* (2013), Sunday *et al.,* (2015).
iv. Example 4 (Highly stiff oscillatory differential equation): As shown in Table 4, the new method outperforms the methods of Sunday *et al.,* (2015) and Okunuga *et al.,* (2013).
v. Example 5 (Highly stiff oscillatory differential equation): Table 5 displays that the new method provide better results compared to the methods of Omar and Adeyeye, (2016) and Skwame *et al.,* (2018).

This study introduces innovative block hybrid method for solving first-order stiff initial value problems, significantly enhancing the numerical analysis. The method demonstrates superior convergence and stability properties compared to existing techniques, offering a more reliable approach for researchers and practitioners. Comprehensive comparative analysis with a range of existing methods highlights the valuable benchmarks for future research. The detailed methodological framework, including rigorous analysis of order, error constant, consistency, zero-stability, and stability regions, serves as a valuable reference for developing similar numerical methods. Practical application to real-life problems validates the methods' effectiveness, as evidenced by superior performance across various examples. In conclusion, the new block method are highly effective for solving first-order stiff initial value problems, exhibiting superior performance in comparison to existing methods in the literature.

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Appendix

The coefficient of $f_n,\ f_{n+\frac{1}{4}},\ f_{n+\frac{1}{2}},\ f_{n+\frac{3}{4}},\ f_{n+1},\ f_{n+\frac{5}{4}},\ f_{n+\frac{3}{2}},\ f_{n+\frac{7}{4}},\ f_{n+2}$ in equation (8) and (9) are given as

$\Omega_{101},\Omega_{102},\Omega_{103},\Omega_{104},\Omega_{105},\Omega_{106},\Omega_{107},\Omega_{108},\Omega_{109}=$
$\frac{1070017h}{14515200},\frac{2233547h}{7257600},\frac{2302297h}{7257600},\frac{2797679h}{7257600},\frac{31457h}{90720},\frac{1573169h}{7257600},\frac{645607h}{7257600},\frac{156437h}{7257600},\frac{33953h}{14515200}$

$\Omega_{111},\Omega_{112},\Omega_{113},\Omega_{114},\Omega_{115},\Omega_{116},\Omega_{117},\Omega_{118},\Omega_{119}=$
$\frac{32377h}{453600},\frac{22823h}{56700},-\frac{21247h}{226800},\frac{15011h}{56700},-\frac{2903h}{11340},\frac{9341h}{56700},-\frac{15577h}{226800},\frac{953h}{56700},-\frac{119h}{64800}$

$\Omega_{121},\Omega_{122},\Omega_{123},\Omega_{124},\Omega_{125},\Omega_{126},\Omega_{127},\Omega_{128},\Omega_{129}=$
$\frac{12881h}{179200},\frac{35451h}{89600},\frac{1719h}{89600},\frac{39967h}{89600},-\frac{351h}{1120},\frac{17217h}{89600},-\frac{7031h}{89600},\frac{243h}{12800},-\frac{369h}{179200}$

$\Omega_{131},\Omega_{132},\Omega_{133},\Omega_{134},\Omega_{135},\Omega_{136},\Omega_{137},\Omega_{138},\Omega_{139}=$
$\frac{4063h}{56700},\frac{5644h}{14175},\frac{61h}{14175},\frac{8188h}{14175},-\frac{454h}{2835},\frac{2308h}{14175},-\frac{989h}{14175},\frac{244h}{14175},-\frac{107h}{56700}$

$\Omega_{141},\Omega_{142},\Omega_{143},\Omega_{144},\Omega_{145},\Omega_{146},\Omega_{147},\Omega_{148},\Omega_{149}=$
$\frac{41705h}{580608},\frac{115075h}{290304},\frac{3775h}{290304},\frac{159175h}{290304},-\frac{125h}{18144},\frac{85465h}{290304},-\frac{24575h}{290304},\frac{5725h}{290304},-\frac{175h}{82944}$

$\Omega_{151},\Omega_{152},\Omega_{153},\Omega_{154},\Omega_{155},\Omega_{156},\Omega_{157},\Omega_{158},\Omega_{159}=$
$\frac{401h}{5600},\frac{279h}{700},\frac{9h}{2800},\frac{403h}{700},-\frac{9h}{140},\frac{333h}{700},\frac{79h}{2800},\frac{9h}{700},-\frac{9h}{5600}$

$\Omega_{161},\Omega_{162},\Omega_{163},\Omega_{164},\Omega_{165},\Omega_{166},\Omega_{167},\Omega_{168},\Omega_{169}=$
$\frac{149527h}{2073600},\frac{408317h}{1036800},\frac{24353h}{1036800},\frac{542969h}{1036800},\frac{343h}{12960},\frac{368039h}{1036800},\frac{261023h}{1036800},\frac{111587h}{1036800},-\frac{8183h}{2073600}$

$\Omega_{171},\Omega_{172},\Omega_{173},\Omega_{174},\Omega_{175},\Omega_{176},\Omega_{177},\Omega_{178},\Omega_{179}=$
$\frac{989h}{14175},\frac{5888h}{14175},\frac{928h}{14175},\frac{10496h}{14175},\frac{908h}{2835},\frac{10496h}{14175},-\frac{928h}{14175},\frac{5888h}{14175},\frac{989h}{14175}$

$\psi_{101},\psi_{102},\psi_{103},\psi_{104},\psi_{105},\psi_{106},\psi_{107},\psi_{108},\psi_{109}=$
$-\frac{1070017h}{14515200},-\frac{2233547h}{7257600},\frac{2302297h}{7257600},-\frac{2797679h}{7257600},\frac{31457h}{90720},-\frac{1573169h}{7257600},\frac{645607h}{7257600},-\frac{156437h}{7257600},\frac{33953h}{14515200}$

$\psi_{111},\psi_{112},\psi_{113},\psi_{114},\psi_{115},\psi_{116},\psi_{117},\psi_{118},\psi_{119}=$
$-\frac{33953h}{14515200},\frac{687797h}{7257600},\frac{1622393h}{7257600},-\frac{876271h}{7257600},\frac{8233h}{90720},-\frac{377521h}{7257600},\frac{147143h}{7257600},-\frac{34453h}{7257600},\frac{7297h}{14515200}$

$\psi_{121},\psi_{122},\psi_{123},\psi_{124},\psi_{125},\psi_{126},\psi_{127},\psi_{128},\psi_{129}=$
$-\frac{119h}{64800},\frac{19937h}{226800},\frac{38149h}{113400},\frac{13739h}{226800},\frac{1513h}{45360},\frac{5581h}{226800},\frac{1189h}{113400},\frac{583h}{226800},\frac{127h}{453600}$

$\psi_{131},\psi_{132},\psi_{133},\psi_{134},\psi_{135},\psi_{136},\psi_{137},\psi_{138},\psi_{139}=$
$\frac{369h}{179200},\frac{8101h}{89600},\frac{28809h}{89600},\frac{17217h}{89600},\frac{209h}{1120},\frac{4833h}{89600},\frac{1719h}{89600},\frac{389h}{89600},\frac{81h}{179200}$

$\psi_{141},\psi_{142},\psi_{143},\psi_{144},\psi_{145},\psi_{146},\psi_{147},\psi_{148},\psi_{149}=$
$-\frac{107h}{56700},\frac{359h}{4050},\frac{4681h}{14175},\frac{2308h}{14175},\frac{1927h}{5670},\frac{2201h}{28350},\frac{61h}{14175},-\frac{26h}{14175},\frac{13h}{56700}$

$\psi_{151},\psi_{152},\psi_{153},\psi_{154},\psi_{155},\psi_{156},\psi_{157},\psi_{158},\psi_{159}=$
$-\frac{175h}{82944},\frac{26365h}{290304},\frac{93025h}{290304},\frac{55225h}{290304},\frac{5125h}{18144},\frac{75175h}{290304},\frac{34015h}{290304},-\frac{2525h}{290304},\frac{425h}{580608}$

$\psi_{161},\psi_{162},\psi_{163},\psi_{164},\psi_{165},\psi_{166},\psi_{167},\psi_{168},\psi_{169}=$
$-\frac{9h}{5600},\frac{241h}{2800},\frac{477h}{1400},\frac{387h}{2800},\frac{209h}{560},\frac{387h}{2800},\frac{477h}{1400},\frac{241h}{2800},-\frac{9h}{5600}$

$\psi_{171},\psi_{172},\psi_{173},\psi_{174},\psi_{175},\psi_{176},\psi_{177},\psi_{178},\psi_{179}=$
$-\frac{8183h}{2073600},\frac{111587h}{1036800},\frac{261023h}{1036800},\frac{368039h}{1036800},\frac{343h}{12960},\frac{542969h}{1036800},\frac{24353h}{1036800},\frac{408317h}{1036800},\frac{149527h}{2073600}$