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Mathematical Modeling of the Dynamics of Psychoactive Drug Abuse with Intervention of Control Agencies

Micah Habila[a], Musa Samuel[b] and James J. Yakoko[a]

[a]*Department of Mathematics, College of Education, Zing, Taraba State, Nigeria.*

[b]*Department of Mathematics, Modibbo Adama University, Yola, Adamawa State, Nigeria.*

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ABSTRACT

Drug abuse continues to pose a significant threat to both developed and developing countries. This paper examines the dynamics of psychoactive drug abuse with intervention of control agencies. The study established that the drug abuse model remains positively invariant over time, as indicated by the positivity of the solutions to the model equations. The study identified both drug abuse endemic and drug abuse free equilibrium states. Basic reproduction number was obtained using the method of next generation matrix. The local stability of the drug abuse free equilibrium state was analyzed using Jacobian method and the analysis showed that the drug abuse free equilibrium state is locally asymptotically stable since all the Eigen values are negative. Similarly, the global stability analysis of the drug abuse free equilibrium state confirmed that the system is globally asymptotically stable. The sensitivity index was computed. The results indicate that parameters with positive sensitivity indices increase the prevalence, while those with negative sensitivity indices decrease it. This suggests that these parameters have the ability to either increase or decrease the persistence of drug abuse in the population. The numerical simulation of the model was performed using MATLAB R2015a software and results show that enlightenment and law enforcement agencies play a vital role in reducing the spread of drug abuse. This research recommends intensification of early enlightenment to curtail the spread of drug abuse. Also making more policies, enactment and enforcement of more laws that will check the use of hard drugs.

1. Introduction

Drug usage encompasses the administration of pharmaceuticals to treat ailments, prevent disease, or promote overall health. However, drug abuse occurs when substances are used non-medicinally in harmful quantities or manners, impacting an individual's physical or mental functions. This harmful use of drugs is distinguished from therapeutic use by the intention and the negative consequences associated with it. According to the National Institute on Drug Abuse (2014), drug abuse is characterized by consumption that deviates from medical guidance, leading to detrimental effects on one's health and well-being.

In contemporary society, drug abuse and addiction have become significant issues, affecting individuals and communities worldwide. Drug abuse is a barrier to modern civilization, potentially hindering individuals from achieving life goals or aspirations. It is associated with various adverse effects, including drowsiness, excitement, insensibility, and negative impacts on mental, socio-economic, and physical health (Hasan & Shahin, 2013). The rising consumption of illegal drugs has serious implications for human and national economic growth. Drugs are profoundly destructive to both the body and mind, with addiction leading to severe health issues and even death. Addiction substantially affects brain structure and function, impairing self-control and decision-making abilities, which exacerbates the challenge of addressing this epidemic. Research indicates that frequent drug abusers benefit from therapy at rehabilitation institutions, which can lead to long-term recovery (Hasan & Shahin, 2013).

* Corresponding author. Tel.: +2347067304465
E-mail address: micahhabila2014@gmail.com (Micah H.)

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Mathematical models have been developed to study the dynamics of drug abuse and inform strategies for intervention. For example, Ahmed, Graupner, and Gutkin (2009) created computational models based on the neurobiology of drug abuse to illustrate addiction's possible contributions. Recent models, such as those by Matonya and Kuznetsov (2021), have demonstrated that incorporating law enforcement and public enlightenment can improve the effectiveness of strategies to combat drug abuse. Globally, drug abuse continues to result in high mortality and disability rates, highlighting the need for enhanced strategies to manage and prevent addiction. In 2012, an estimated 183,000 drug-related fatalities were recorded, with between 162 million and 324 million people aged 15 to 64 having used an illegal substance that year (UNODC, 2014). Drug abuse is linked to 50% of mental disease cases and causes numerous deaths (Canadian Mental Health Association, 2005).

Elbaz and El-Awady, (2023) explores the dynamics of soft drug epidemics, particularly how the spread of drug use can be either extinguished or sustained within a population. Their model examines the sensitivity of the epidemic to various parameters, which is crucial in understanding the conditions under which drug use persists or dies out. The study identifies specific conditions that can lead to the extinction of drug use within a population, meaning that drug use eventually diminishes and disappears. On the other hand, they also determine scenarios where drug use persists and becomes an endemic issue. Sensitivity analysis in the model shows how different intervention strategies, such as law enforcement or public health initiatives, can alter the course of the drug epidemic. Effective interventions could tip the balance towards the extinction of drug use.

Magfirah, *et al*. (2023) focuses on the dynamics of drug addiction within South Sulawesi Province. The study uses a mathematical model to simulate how drug abuse spreads and how it can be controlled through various interventions. The model highlights the unique factors contributing to drug abuse in South Sulawesi, including social, economic, and environmental influences. The research emphasizes the importance of targeted interventions, such as rehabilitation programs and community awareness campaigns, in controlling the spread of drug addiction. The findings suggest that a combination of these strategies can significantly reduce the prevalence of drug abuse in the region.

Anggriani, Toaha and Kasbawati (2021) investigate the optimal control of mathematical models related to the spread of drug abuse. The authors aim to determine the most effective strategies for managing drug abuse through interventions by control agencies. The study presents optimal control strategies that can be employed by law enforcement and public health agencies to minimize drug abuse. These strategies include early detection, quarantine, and treatment programs. The findings suggest that well-coordinated interventions can lead to a significant reduction in drug abuse rates. The model highlights the importance of continuous monitoring and adaptation of strategies to address the evolving nature of drug abuse. Efforts to reproduce the complexities of drug use in communities are often hindered by abusers. However, continued research and improved intervention models can help mitigate the global burden of drug abuse. However, it was observed that their work would have given a better result if they had incorporated law enforcement agencies and mass enlightenment. This study presents a modified mathematical model that study the dynamics of drug abuse by incorporating enlightened susceptible individuals, law enforcement agencies and a detention center in an attempt to improve on the model presented by Matonya and Kuznetsov (2021)

2. Formulation of model

In this section, we analyze the dynamics of drug user populations by dividing the human population into eight (8) subgroups and an independent compartment law enforcement agency. Each subgroup represents a different stage or state in relation to drug use, and the interactions between these subgroups determine the overall dynamics of drug use in the population. We consider the following nine subpopulations:

1) **unenlightened Susceptible Individuals (S_u):** These are individuals who have not yet started using drugs but are at risk of doing so if they come into contact with drug users. They are referred to as "susceptible" because their exposure to drug users may lead them to begin using drugs.

2) **Enlightened Susceptible Individuals (S_e):** These individuals are aware of the risks associated with drug use but remain in environments where drug use is prevalent. Although they are informed, they are still susceptible to starting drug use due to their surroundings.

3) **Exposed Individuals (E):** This subgroup includes individuals who have begun using drugs but have not yet exhibited symptoms or been classified as either light or heavy drug users. They are in the early stages of drug use.

4) **Light Drug Users (I_1):** These individuals have started using drugs and exhibit symptoms of drug use. They are capable of influencing vulnerable individuals to start using drugs.

5) **Heavy Drug Users (I_2):** This subgroup consists of individuals with severe drug addiction. They display significant symptoms and have a higher likelihood of encouraging others to start using drugs.

6) **Detained Individuals ($D.$):** Heavy drug users who have been detained by law enforcement agencies fall into this category. Their detention is a form of intervention to prevent further drug use.

7) **Patients in Rehabilitation (Q):** These are individuals who have entered rehabilitation centers for treatment. They are in the process of recovering from drug addiction.

8) **Recovered Individuals (R):** Individuals who have successfully completed treatment or have self-reflected and stopped using drugs. They are temporarily invulnerable to drug use but may become exposed again in the future.

9) **Law enforcement Agency** (L): Government agency who act in an organized manner to enforce the law by discovering, investigating, deterring, rehabilitating, or punishing people who violate the rules and norms governing that society.

Interactions between Subpopulations

Transition to Exposed (E): Susceptible individuals (S_u) and (S_e) may become exposed (E) if they interact with light or heavy drug users (I_1 or I_2). Close relationships or frequent contact between susceptible humans and drug users enhance the odds of becoming a drug user and being in a high risk of exposure at a rate β . A proportion of $(1-\theta)$ enlightened susceptible (S_e) individuals is transmitted to (E) individuals at the rate $\lambda.$ Extensive exposure to a sensitive drug use setting results in light drug users I_1 or heavy drug user I_2 at a rate α_1 and proportion of $(1-\rho$) or ρ respectively.

Exposed (E) Individuals: Exposed individuals (E) may progress to becoming light drug users (I_1) or heavy drug users (I_2), depending on the extent of their drug use.

Progression from Light to Heavy Drug Use: Light drug users (I_1) may develop a stronger addiction over time and become heavy drug users (I_2). Low drug users may develop hook to drugs and become a heavy drug user over time at a rate $\rho_1\sigma_1$. Also, low Drug users may recover and become temporarily invulnerable at a rate $\rho_3\sigma_1$ through self-reflection or family intervention. Otherwise, depending on the character of the family and the environment, the low drug users progress to rehabilitation facilities for treatment at the rate $\rho_2\sigma_1$. The same low drug users in a proportion $\psi_3 L$ due to interaction with law enforcement agencies return to exposed individuals. Heavy drug abusers may be transported to rehabilitation clinics for treatment at the rate σ_2 ; otherwise, if not treated, they die as a result of substantial drug addiction at a rate μ_2 .

Recovery and Rehabilitation: Light drug users (I_1) may recover and move to the recovered state (R) through self-reflection, family intervention, or entry into rehabilitation facilities (Q). Successful treated drug users in rehabilitation clinics recover and move to the recovery class at a rate α_2 .

Law Enforcement Interaction: A proportion of heavy drug abusers as a result of interaction with law enforcement agencies may also be transported to detention centers at a rate $\psi_4 L$. The drug addicts under detention move to rehabilitation center at the rate π.

Recovery and Relapse: Recovered individuals (R) are temporarily invulnerable but may relapse and return to the exposed state (E) at a rate α_3 over time. The likelihood of relapse depends on various factors, including continued exposure to drug-using environments.

Recruitment and Natural Death: The population is constantly replenished with new individuals at steady rates Λ_1 and Λ_2, while natural death occurs at a rate μ_1 and μ_2 .

Control Efforts: A control coefficient ω represents the efforts to detect and quarantine substance users early, aiming to reduce the rate at which individuals are exposed to drug use.

The schematic diagram and equations for the dynamics of illicit Drugs is presented in figure 1 and system (1)

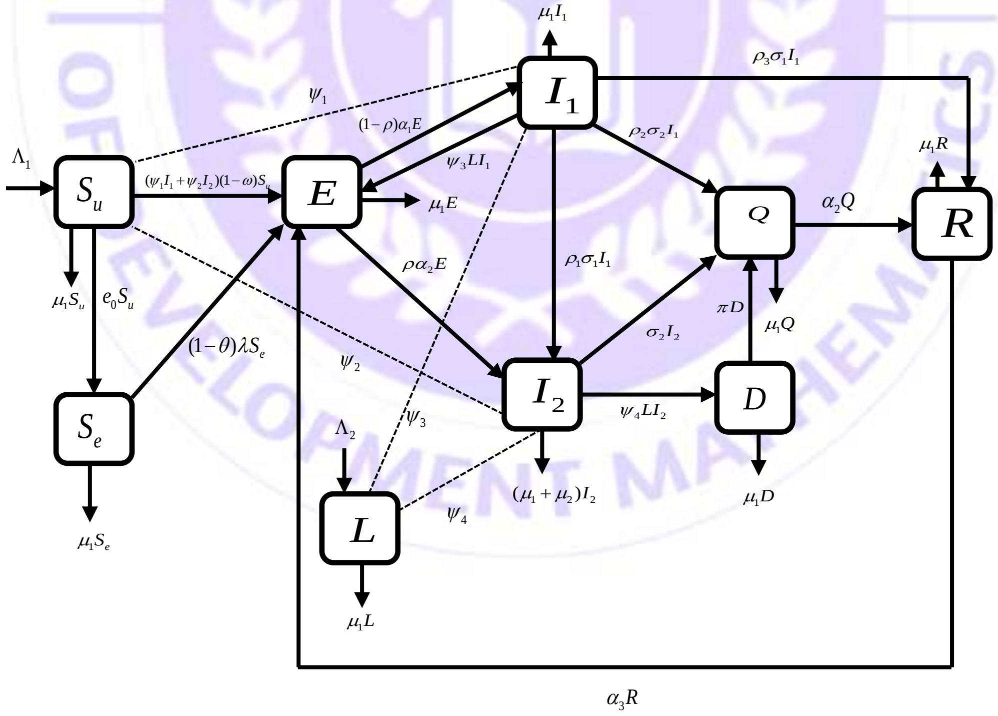


Figure 1: Schematic model diagram

$$\left.\begin{aligned}
\dot{S}_u &= \Lambda_1 - e_0 S_u - (\psi_1 I_1 + \psi_2 I_2)(1-\omega)S_u - \mu_1 S_u \\
\dot{S}_e &= e_0 S_u - (1-\theta)\lambda S_e - \mu_1 S_e \\
\dot{E} &= (\psi_1 I_1 + \psi_2 I_2)(1-\omega)S_u + (1-\theta)\lambda S_e + \psi_3 L I_1 + \alpha_3 R - \left(\alpha_1 + \mu_1\right)E \\
\dot{I}_1 &= (1-\rho)\alpha_1 E - (\rho_1 + \rho_2 + \rho_3 + \psi_3 L + \mu_1)I_1 \\
\dot{I}_2 &= \rho\alpha_1 E + \rho_1\sigma_1 I_1 - (\psi_4 L + \sigma_2 + \mu_1 + \mu_2)I_2 \\
\dot{D} &= \psi_4 L I_2 - (\pi + \mu_1)D \\
\dot{Q} &= \rho_2\sigma_1 I_1 + \sigma_2 I_2 + \pi D - (\alpha_2 + \mu_1)Q \\
\dot{R} &= \alpha_2 Q + \rho_3\sigma_1 I_1 - (\alpha_3 + \mu_1)R \\
\dot{L} &= \Lambda_2 - \mu_1 L
\end{aligned}\right\} \tag{1}$$

with initial conditions:

$$\begin{cases} S_u(0) > 0, S_e(0) \geq 0, E(0) \geq 0, I_1(0) \geq 0, \\ I_2(0) \geq 0, D(0) \geq 0, Q(0) \geq 0, R(0) \geq 0. \end{cases} \tag{2}$$

3. Analysis of Drug Abuse Model

In this section, we prove the positivity and boundedness of the solutions of drug abuse model, established the equilibrium points and analyze the model for stability.

3.1 Positivity of the solutions of drug abuse model

With initial condition $S_u(0) \geq 0,\ S_e(0) \geq 0,\ E(0) \geq 0,\ I_1(0) \geq 0,\ I_2(0) \geq 0,\ D(0) \geq 0,\ Q(0) \geq 0,$ $R(0) \geq 0,$ all the parameters are non-negative constants with their biological interpretations. Each of the total subpopulation $N_1(t)$ and $N_2(t)$ is assumed to be non-negative at $t > 0$.

Theorem 1

The solution set $\{S_u, S_e, E, I_1, I_2, D, Q, R, L\}$ of the model equations (1) with non-negative initial conditions remain non-negative at $t > 0$. Hence, the initial set will be:

$$\Omega = \left\{\left(S_u(0), S_e(0), E(0), I_1(0), I_2(0), D(0), Q(0), R(0), L(0)\right) \in R_+^{\,9}\right\} \ \forall \text{ time } t$$

Proof

Given the initial conditions of the model equations (1) with $S_u(0),\ S_e(0),\ E(0),\ I_1(0),\ I_2(0),\ D(0),$ $Q(0),\ R(0),\ L(0)$ are non-negative. Then, the equations of the system with only term related to the state variables are considered. Thus, rewriting equation (1) to obtain;

$$\left.\begin{aligned}
\dot{S}_u &\geq -(e_0 + (\psi_1 I_1 + \psi_2 I_2)\beta_1 + \mu_1)S_u \\
\dot{S}_e &\geq -(\lambda + \mu_1)\beta_2 S_e \\
\dot{E} &\geq -(\alpha_1 + \mu_1)E \\
\dot{I}_1 &\geq -(\rho_1 + \rho_2 + \rho_3 + \psi_3 L + \mu_1)I_1 \\
\dot{I}_2 &\geq -(\psi_4 L + \sigma_2 + \mu_1 + \mu_2)I_2 \\
\dot{D} &\geq -(\pi + \mu_1)D \\
\dot{Q} &\geq -(\alpha_2 + \mu_1)Q \\
\dot{R} &\geq -(\alpha_3 + \mu_1)R \\
\dot{L} &\geq -\mu_1 L
\end{aligned}\right\} \tag{3}$$

Note: $\beta_1 = (1-\omega),\ \beta_2 = (1-\theta)$

Solving (3) using separation of variables we obtain (4) and (5):

$$S_u(t) \geq S_u(0)e^{-\int_0^t (\psi_1 I_1(y)+\psi_2 I_2(y))\beta_1 dy + (e_0+\mu_1)t} \qquad \forall t > 0 \tag{4}$$

$$S_e(t) \geq S_e(0)e^{-(\lambda+\mu_1)\beta_2 t} \qquad \forall t > 0 \tag{5}$$

In order to show that $E(t), I_1(t), I_2(t), D(t), Q(t), R(t), L(t) > 0$ remain non-negative for all instant t > 0, the study followed similar approach. Thus, the remaining model (1) variables remain non-negative for all instant $t > 0$.

3.2 Boundedness of the solution of drug abuse model

The model equation (1) is epidemiologically feasible.

Theorem 2

The solution set $\{S_u, S_e, E, I_1, I_2, D, Q, R, L\}$ of the model equations (1) is assumed to be non-negative at $t > 0$. using standard comparison theorem; hence, all solutions to the system of equations (1) are non-negative. The region

$$\Omega = \{S_u(t), S_e(t), E(t), I_1(t), I_2(t), D(t), Q(t), R(t), L(t) \in \mathbb{R}_+^{\,9} :$$
$$\left. 0 < S_u(t) + S_e(t) + E(t) + I_1(t) + I_2(t) + D(t) + Q(t) + R(t) \leq \frac{\Lambda_1}{\mu_1}, 0 < L(t) \leq \frac{\Lambda_2}{\mu_1} \right\}$$

is positively invariant for the system (1) for all time t.

Proof

The total population in (1), yields

$$\frac{dN_1}{dt}=\frac{dS_u}{dt}+\frac{dS_e}{dt}+\frac{dE}{dt}+\frac{dI_1}{dt}+\frac{dI_2}{dt}+\frac{dD}{dt}+\frac{dQ}{dt}+\frac{dR}{dt}$$

$$\left.\begin{aligned}\frac{dN_1}{dt}&\le\Lambda_1-\mu_1N_1\\\frac{dN_2}{dt}&\le\Lambda_2-\mu_1N_2\end{aligned}\right\}\qquad(6)$$

On solving (6), the following are obtained

$$N_1\le\frac{\Lambda_1}{\mu_1}\qquad(7)$$

Also, following the same approach for law enforcement agencies, we obtain,

$$N_2\le\frac{\Lambda_2}{\mu_1}\qquad(8)$$

Thus, the feasible solution set of Drug abuse model (1) is positively invariant and mathematically well posed in the region Ω. Therefore, it is sufficient to study the dynamic of the model in Ω.

Table 1. Drug abuse model parameters description

Parameter	**Description**
Λ_1 and Λ_2	The recruitment rates of S_u and L respectively
ψ_1 and ψ_2	The rate that S become E due to I_1 and I_2 respectively
α_1	Progression rate out of E due to drug users states
ρ	Proportion of E becoming I_2
σ_1	Rate at which individuals leave I_1
ρ_1	Proportion of I_1 becoming I_2
ρ_2	Proportion of I_1 becoming Q
σ_2	Rate at which I_2 become Q
e_0	Enlightenment Rate that result to s_e
ρ_3	Proportion of I_1 becoming R
μ_1	Natural death rate of human beings
μ_2	Heavy drug induced death
ω	Control efforts to protect S_u
α_3	Rate at which R Returns to E after recovery
θ	The measure of effectiveness of enlightenment on S_e
λ	Transmission rate of S_e to E
π	Rate at which D becomes Q

ψ_3	Rate of interaction between L and I_1
ψ_4	Rate of interaction between L and I_2
N	Total human population

Table 2. Drug abuse model variables

Variable	**Description**
$S_u(t)$	Unenlightened Susceptible Humans at time t .
$S_e(t)$	Enlightened Susceptible Humans at time t .
$E(t)$	Exposed drug users at time t .
$I_1(t)$	Light drug addicted users at time t .
$I_2(t)$	Heavy drug addicted users at time t .
$D(t)$	Heavy Drug Users in Detention center at time t .
$Q(t)$	Drug users under rehabilitation at time t .
$R(t)$	Recovered drug users after treatment at time t .
$L(t)$	Law Enforcement Agencies at time t .

3.3 Equilibrium states of Drug abuse model

In this section, we established equilibrium points of the drug-abuse model given by (1)

3.3.1 Drug abuse free equilibrium state

To obtain Drug abuse equilibrium states we set:

$$\frac{dS_u}{dt} = \frac{dS_e}{dt} = \frac{dE}{dt} = \frac{dI_1}{dt} = \frac{dI_2}{dt} = \frac{dD}{dt} = \frac{dQ}{dt} = \frac{dR}{dt} = \frac{dL}{dt} = 0 \tag{9}$$

To obtain Drug abuse free equilibrium state (E^0) of the model equations (1), we set:

$$E = I_1 = I_2 = D = Q = R = 0 \tag{10}$$

Thus, the Drug abuse free equilibrium state denoted by:

$E^0 = \left(S_u{}^0, S_e{}^0, E^0, I_1{}^0, I_2{}^0, D^0, Q^0, R^0, L^0\right)$ is given as

$$E^0 = \left(\frac{\Lambda_1}{e_0 + \mu_1}, \frac{e_0 \Lambda_1}{((1-\theta)\lambda - \mu_1)(e_0 + \mu_1)}, 0, 0, 0, 0, 0, 0, \frac{\Lambda_2}{\mu_1} \right) \tag{11}$$

3.3.2 Drug abuse present equilibrium state

We also obtained Drug abuse present equilibrium state (E^*) of the model equations (1) by setting:

$$E \neq I_1 \neq I_2 \neq D \neq Q \neq R \neq 0 \tag{12}$$

Thus, the Drug abuse present equilibrium state is denoted by:

$$S_u^* = \frac{\Lambda_1(e_0+\mu_1)\beta_2\lambda + \mu_1\left[\left(\alpha_1+\mu_1\right)\mu_1 + (\psi_3\Lambda_2\tau_1+\alpha_3\mu_1\tau_5)\left(e_0+\mu_1+(\psi_1\tau_1+\psi_2\tau_2)\beta_1\right)\right]}{(e_0+\mu_1)+(\psi_1\tau_1+\psi_2\tau_2)\beta\left(\beta_2 e_0\Lambda_1\mu_1\right)\left(e_0+\mu_1+(\psi_1\tau_1+\psi_2\tau_2)\beta_1\right)+\left((e_0+\mu_1)\beta_2\lambda+\mu_1\right)\left(\beta_1\mu_1\Lambda_1(\psi_1\tau_1+\psi_2\tau_2)\right)} \times \frac{1}{\tau_6} \tag{13}$$

$$S_e^* = \frac{\Lambda_1(e_0+\mu_1)\beta_2\lambda + \mu_1\left[\left(\alpha_1+\mu_1\right)\mu_1 + (\psi_3\Lambda_2\tau_1+\alpha_3\mu_1\tau_5)\left(e_0+\mu_1+(\psi_1\tau_1+\psi_2\tau_2)\beta_1\right)\right]}{(e_0+\mu_1)+(\psi_1\tau_1+\psi_2\tau_2)\beta\left(\beta_2 e_0\Lambda_1\mu_1\right)\left(e_0+\mu_1+(\psi_1\tau_1+\psi_2\tau_2)\beta_1\right)+\left((e_0+\mu_1)\beta_2\lambda+\mu_1\right)\left(\beta_1\mu_1\Lambda_1(\psi_1\tau_1+\psi_2\tau_2)\right)} \tau_6 \tag{14}$$

$$I_1^* = \frac{\mu_1(1-\rho)\alpha_1}{\mu_1(\rho_1+\rho_2+\rho_3)+\psi_3\Lambda_2+\mu_1^2}\tau_6 \tag{15}$$

$$I_2^* = \frac{(\mu_1\rho\alpha_1(\mu_1(\rho_1+\rho_2+\rho_3)+\psi_4\Lambda_2+\mu_1^2)+\mu_1\rho_1\sigma_1(\mu_1(1-\rho)\alpha_1}{\mu_1(\rho_1+\rho_2+\rho_3)+\psi_4\Lambda_2+\mu_1^2)(\psi_4\Lambda_2+\mu_1(\sigma_2+\mu_1+\mu_2))}\tau_6 \tag{16}$$

$$D^* = \frac{\psi_4\Lambda_2}{\mu_1(\pi+\mu_1)}\left(\frac{(\mu_1\rho\alpha_1(\mu_1(\rho_1+\rho_2+\rho_3)+\psi_4\Lambda_2+\mu_1^2)+\mu_1\rho_1\sigma_1(\mu_1(1-\rho)\alpha_1}{\mu_1(\rho_1+\rho_2+\rho_3)+\psi_4\Lambda_2+\mu_1^2)(\psi_4\Lambda_2+\mu_1(\sigma_2+\mu_1+\mu_2))}\right)\tau_6 \tag{17}$$

$$Q^* = \frac{[\rho_2\sigma_1\tau_1(\mu_1(\pi+\mu_1))+\sigma_2\tau_2(\mu_1(\pi+\mu_1))+\pi\tau_2(\psi_4\Lambda_2)]}{(\alpha_2+\mu_1)(\mu_1(\pi+\mu_1))}\tau_6 \tag{18}$$

$$R^* = \left(\frac{\alpha_2\tau_4+\rho_3\sigma_1\tau_1}{(\alpha_3+\mu_1)}\right)\tau_6 \tag{19}$$

Therefore, the drug abuse endemic equilibrium state (E^*) is given by:

$$E^* = \left(S_u^*, S_e^*, E^*, I_1^*, I_2^*, D^*, Q^*, R^*, L^*\right) \tag{20}$$

where:

$$\tau_3 = \frac{\psi_4\Lambda_2}{\mu_1(\pi+\mu_1)}$$

$$\tau_1 = \frac{\mu_1(1-\rho)\alpha_1}{\mu_1(\rho_1+\rho_2+\rho_3)+\psi_3\Lambda_2+\mu_1^2}$$

$$\tau_2 = \frac{(\mu_1\rho\alpha_1(\mu_1(\rho_1+\rho_2+\rho_3)+\psi_4\Lambda_2+\mu_1^2)+\mu_1\rho_1\sigma_1(\mu_1(1-\rho)\alpha_1}{\mu_1(\rho_1+\rho_2+\rho_3)+\psi_4\Lambda_2+\mu_1^2)(\psi_4\Lambda_2+\mu_1(\sigma_2+\mu_1+\mu_2))}$$

$$\tau_5 = \left(\frac{\alpha_2\tau_4 + \rho_3\sigma_1\tau_1}{(\alpha_3 + \mu_1)} \right)$$

$$\tau_6 = \frac{\left(\beta_2 e_0 \Lambda_1 \mu_1\right)\left(e_0 + \mu_1 + (\psi_1\tau_1 + \psi_2\tau_2)\beta_1\right) + \left((e_0 + \mu_1)\beta_2\lambda + \mu_1\right)\left(\beta_1\mu_1\Lambda_1(\psi_1\tau_1 + \psi_2\tau_2)\right)}{(e_0 + \mu_1)\beta_2\lambda + \mu_1\left[\left(\alpha_1 + \mu_1\right)\mu_1 + (\psi_3\Lambda_2\tau_1 + \alpha_3\mu_1\tau_5)\left(e_0 + \mu_1 + (\psi_1\tau_1 + \psi_2\tau_2)\beta_1\right)\right]}$$

3.4 Reproduction Number

Recall that the number of secondary infection cases brought on by the introduction of a single infectious agent into a population that is entirely susceptible is the basic reproduction number. We opt for a generalized approach known as the Next Generation Matrix technique since the suggested mathematical model for subpopulations with different susceptibilities to infection to find the basic reproduction number R_0 (Diekmann and Heesterbeek, 2000).The threshold R_0 is represented by $R_0 = \rho(FV^{-1})$
where ρ is the spectral radius.

Given the infectious classes as E, I_1, I_2 in (1), we can create a vector F that represents the new infections. The components of the vector F are obtained by considering the terms denoting new infections from the susceptible entering the exposed compartment:

$$F = \begin{pmatrix} 0 & \phi_1 & \phi_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{21}$$

where:

$$\phi_1 = \frac{\psi_1\beta_1\Lambda_1}{e_0 + \mu_1}, \ \phi_2 = \frac{\psi_2\beta_1\Lambda_1}{e_0 + \mu_1}$$

The study also consider V which denotes the net transfers between the infectious compartments in (1) which are given by

$$V = \begin{pmatrix} -M_1 & -K_1 & 0 \\ K_2 & -M_2 & 0 \\ K_3 & K_4 & -M_3 \end{pmatrix} \tag{22}$$

where

$$M_1 = (\alpha_1 + \mu_1)$$
$$M_2 = \left(\rho_1 + \rho_2 + \rho_3 + \frac{\psi_3 \Lambda_2}{\mu_1} + \mu_1 \right)$$
$$M_3 = \left(\frac{\psi_4 \Lambda_2}{\mu_1} + \sigma_2 + \mu_1 + \mu_2 \right)$$

$$K_1 = \frac{\psi_3 \Lambda_2}{\mu_1}$$
$$K_2 = (1-\rho)\alpha_1$$
$$K_3 = \rho\alpha_1$$
$$K_4 = \rho_1\sigma_1$$

Therefore, the basic reproduction number of the violence only model (1), denoted by $R_0 = \rho(FV^{-1})$ is given by

$$R_0 = \frac{K_2\left(\phi_2 K_4 + \phi_1 M_3\right) + \phi_2 K_3 M_2}{M_3\left(K_1 K_2 - M_1 M_2\right)} \qquad 23)$$

3.5 Local Stability of Drug Abuse Free Equilibrium State

Theorem 3

The Drug abuse free equilibrium state of the model equations (1) is Locally and Asymptotically Stable (LAS) if $\lambda_i < 1 \quad \forall \ i = 1,2,3,4,5,6,7,8$ otherwise unstable.

Proof

At drug abuse free equilibrium, we computed and assessed the eigen values of Jacobian matrix in order to verify the drug free stationary point whether it is locally and asymptotically stable. Furthermore showed that the real parts of the eigen values of the matrix are negative. The Jacobian matrix at drug abuse free equilibrium state is given as:

$$J_0 = \begin{pmatrix} -a_1 & 0 & 0 & a_2 & a_3 & 0 & 0 & 0 \\ e_0 & -a_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_2\lambda & -a_5 & a_6 & a_7 & 0 & 0 & \alpha_3 \\ 0 & 0 & a_8 & -a_9 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho\alpha_1 & \rho_1\sigma_1 & -a_{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{11} & -a_{12} & 0 & 0 \\ 0 & 0 & 0 & \rho_2\sigma_1 & \sigma_2 & \pi & -a_{13} & 0 \\ 0 & 0 & 0 & \rho_3\sigma_1 & 0 & 0 & \alpha_2 & -a_{14} \end{pmatrix} \qquad (24)$$

where:

$$a_1 = (e_0 + \mu_1),\ a_2 = -\frac{\psi_1\beta_1\Lambda_1}{e_0 + \mu_1}\ a_3 = -\frac{\psi_2\beta_1\Lambda_1}{e_0 + \mu_1}\ a_4 = \left(\beta_2\lambda + \mu_1\right)$$

$$a_5 = (\alpha_1 + \mu_1)\ a_6 = \frac{\psi_1\beta_1\Lambda_1}{e_0 + \mu_1}\ a_7 = \frac{\psi_2\beta_1\Lambda_1}{e_0 + \mu_1}\ a_8 = (1-\rho)\alpha_1$$

$$a_9 = (\rho_1 + \rho_2 + \rho_3 + \psi_3 \frac{\Lambda_2}{\mu_1} + \mu_1) \quad a_{10} = (\frac{\psi_4 \Lambda_2}{\mu_1} + \sigma_2 + \mu_1 + \mu_2) \quad a_{11} = \frac{\psi_4 \Lambda_2}{\mu_1}$$

$$a_{12} = (\pi + \mu_1)\,, a_{13} = (\alpha_2 + \mu_1) \quad a_{14} = (\alpha_3 + \mu_1)$$

Equation (24) was later reduced to upper triangular matrix as:

$$|J_0| = \begin{vmatrix} -a_1 & 0 & 0 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & -a_4 & 0 & -\frac{e_0 a_2}{a_1} & -\frac{e_0 a_3}{a_1} & 0 & 0 & 0 \\ 0 & 0 & -a_5 & T & T_2 & 0 & 0 & \alpha_3 \\ 0 & 0 & 0 & -T_3 & T_4 & 0 & 0 & -\frac{a_8 \alpha_3}{a_5} \\ 0 & 0 & 0 & 0 & -T_5 & 0 & 0 & T_6 \\ 0 & 0 & 0 & 0 & 0 & -a_{12} & 0 & T_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_{13} & T_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_9 \end{vmatrix} \tag{25}$$

where:

$$T_3 = -\frac{a_1 a_8 \beta_2 e_0 \lambda + a_1 a_4 a_5 a_9 - a_1 a_4 a_6 a_8}{a_5 a_4 a_1}$$

$$T_5 = \frac{-a_1 a_{10} a_4 a_6 a_8 + a_1 a_{10} a_4 a_5 a_9 + a_{10} a_2 a_8 \beta_2 e_0 \lambda - a_1 a_4 a_7 a_9 \rho \alpha_1 + a_3 a_9 \beta_2 e_0 \lambda \rho \alpha_1 + a_8 a_3 \beta_2 e_0 \lambda \rho_1 \sigma_1 - a_8 a_1 a_4 a_7 \rho_1 \sigma_1}{a_2 a_8 \beta_2 e_0 \lambda + a_1 a_4 a_5 a_9 - a_1 a_4 a_6 a_8}$$

$$T_9 = \frac{\begin{array}{l} -a_{13} a_{12} (a_1 a_{10} a_4 a_8 \rho_3 \alpha_3 \sigma_3 + a_1 a_{14} a_4 a_7 a_8 \rho_1 \sigma_1 + a_1 a_{14} a_4 a_7 a_9 \rho \alpha_1 + a_1 a_{10} a_{14} a_4 a_5 a_9 + a_1 a_{10} a_{14} a_4 a_6 a_8 - \\ a_{14} a_3 a_8 \beta_2 e_0 \lambda \rho_1 \sigma_1 - a_{14} a_3 a_9 \beta_2 e_0 \lambda \rho \alpha_1 - a_{10} a_{14} a_2 a_8 \beta_2 e_0 \lambda) - \alpha_2 \alpha_3 a_1 \alpha_4 (a_{10} a_{12} a_8 \rho_2 \sigma_1 + a_{11} a_8 \pi \rho_1 \sigma_1 \\ + a_{11} a_9 \pi \rho \alpha_1 + a_{12} a_8 \rho_1 \sigma_1 \sigma_2 + a_{12} a_9 \rho \alpha_1 \sigma_2) \end{array}}{\begin{array}{l} a_{13} a_{12} (-a_1 a_{10} a_4 a_6 a_8 + a_1 a_{10} a_{14} a_4 a_5 a_9 + a_{10} a_2 a_8 \beta_2 e_0 \lambda - a_1 a_4 a_7 a_9 \rho \alpha_1 + a_3 a_9 \beta_2 e_0 \lambda \rho \alpha_1 + \\ a_3 a_8 \beta_2 e_0 \lambda \rho_1 \sigma_1 + a_1 a_4 a_7 a_8 \rho_1 \sigma_1) \end{array}}$$

$$|J - \lambda I| = 0$$

$$\begin{vmatrix} -a_1-\lambda_1 & 0 & 0 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & -a_4-\lambda_2 & 0 & -\frac{e_0 a_2}{a_1} & -\frac{e_0 a_3}{a_1} & 0 & 0 & 0 \\ 0 & 0 & -a_5-\lambda_3 & T & T_2 & 0 & 0 & \alpha_3 \\ 0 & 0 & 0 & -T_3-\lambda_4 & T_4 & 0 & 0 & -\frac{a_8\alpha_3}{a_5} \\ 0 & 0 & 0 & 0 & -T_5-\lambda_5 & 0 & 0 & T_6 \\ 0 & 0 & 0 & 0 & 0 & -a_{12}-\lambda_6 & 0 & T_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_{13}-\lambda_7 & T_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_9-\lambda_8 \end{vmatrix} = 0 \tag{26}$$

From (26) we obtain the Eigen values as follows:

$$\left[(-a_1-\lambda_1)(-a_4-\lambda_2)(-a_5-\lambda_3)(-T_3-\lambda_4)(-T_5-\lambda_5)(-a_{12}-\lambda_6)(-a_{13}-\lambda_7)(-T_9-\lambda_8)\right]=0 \tag{27}$$

$$\lambda_1 = -a_1 = -\left(e_0+\mu_1\right)$$

$$\lambda_1 < 0$$

$$\lambda_2 = -a_4 = -\left(\beta_2\lambda+\mu_1\right)$$

$$\lambda_2 < 0$$

$$\lambda_3 = -a_5 = -(\alpha_1+\mu_1)$$

$$\lambda_3 < 0$$

$$\lambda_4 = -T_3 = -\frac{a_1a_8\beta_2e_0\lambda + a_1a_4a_5a_9 - a_1a_4a_6a_8}{a_5a_4a_1}$$

$$\Rightarrow \frac{a_1a_8\beta_2e_0\lambda + a_1a_4a_5a_9}{a_1a_4a_6a_8} < 1$$

Since

$$-T_5 - \lambda_5 = 0$$

$$\lambda_5 = -T_5 = -\frac{a_1a_{10}a_4a_6a_8 + a_1a_{10}a_4a_5a_9 + a_{10}a_2a_8\beta_2e_0\lambda - a_1a_4a_7a_9\rho\alpha_1 + a_3a_9\beta_2e_0\lambda\rho\alpha_1 + a_8a_3\beta_2e_0\lambda\rho_1\sigma_1 - a_8a_1a_4a_7\rho_1\sigma_1}{a_2a_8\beta_2e_0\lambda + a_1a_4a_5a_9 - a_1a_4a_6a_8}$$

$$\lambda_5 < 0$$

If

$$a_1a_{10}a_4a_6a_8 + a_1a_{10}a_4a_5a_9 + a_{10}a_2a_8\beta_2e_0\lambda + a_3a_9\beta_2e_0\lambda\rho\alpha_1 - a_1a_4a_7a_9\rho\alpha_1 + a_8a_3\beta_2e_0\lambda\rho_1\sigma_1 - a_8a_1a_4a_7\rho_1\sigma_1 < 0$$

$$a_1a_{10}a_4a_6a_8 + a_1a_{10}a_4a_5a_9 + a_{10}a_2a_8\beta_2e_0\lambda + a_3a_9\beta_2e_0\lambda\rho\alpha_1 + a_8a_3\beta_2e_0\lambda\rho_1\sigma_1 < a_1a_4a_7a_9\rho\alpha_1 + a_8a_1a_4a_7\rho_1\sigma_1$$

$$\Rightarrow \frac{a_1a_{10}a_4a_6a_8 + a_1a_{10}a_4a_5a_9 + a_{10}a_2a_8\beta_2e_0\lambda + a_3a_9\beta_2e_0\lambda\rho\alpha_1 + a_8a_3\beta_2e_0\lambda\rho_1\sigma_1}{a_1a_4a_7(a_9\rho\alpha_1 + a_8\rho_1\sigma_1)} < 1$$

Since

$$-a_{12} - \lambda_6 = 0$$

$$\lambda_6 = -a_{12} = -(\pi+\mu_1)$$

then

$$\lambda_6 < 0$$
$$\lambda_7 = -a_{13}$$
$$\lambda_7 = -(\alpha_2 + \mu_1)$$

Then

$$\lambda_7 < 0$$

and

$$-T_9 - \lambda_8 = 0$$

for $\lambda_8 < 0$

Since all the possibilities for $\lambda_i, \mathrm{i} = 1 \ldots 8$ hold, it implies

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 < 0 \tag{28}$$

Hence the negativity requirement for local stability is satisfied.

3.6 Global Stability of Drug Abuse Free Equilibrium State

For the system to be global asymptotic stability of drug abuse free equilibrium state we utilized Castillo-Chavez, Feng, and Huang (2002) approach. Hence, we re-write equations (1) as follows

$$\frac{dX}{dt} = H(X, Z)$$
$$\frac{dZ}{dt} = G(X, Z), G(X, Z) = 0$$

where:

$X \in \mathbb{R}^6$ and $X = (S_u, S_e, D, Q, R, L)$ representing the number of uninfected individuals and

$Z \in \mathbb{R}^3$, $Z = (E, I_1, I_2)$ representing the number of infected individuals.

The disease free equilibrium of the system now becomes $E^0 = (X^*, 0)$. The global asymptotic stability of the drug abuse free steady state is guaranteed if the following conditions are satisfied.

$(a) \dfrac{dX}{dt} = H(X, 0)$, X^* is Globally Asymptotically Stable (G.A.S)

$(b) \hat{G}(X, Z) = PZ - G(X, Z) \geq 0$ for $(X, Z) \in \Omega$

Here $P = D_z G(X, 0)$ is an M-matrix and Ω is the invariant region of the model.

Theorem 4

The Drug abuse free equilibrium state of the model equation (1) is Globally and Asymptotically Stable (GAS) if $R_0 < 1$ and conditions (a) and (b) above are satisfied.

Proof

We begin by establishing that the conditions (a) and (b) hold when $R_0 < 1$. For the drug-abuse population, we have

$$Z=(E,I_1,I_2)^T \tag{29}$$

$$H(X,0)=\begin{pmatrix}\Lambda_1-(e_0+\mu_1)S_u\\ e_0S_u-(\beta_2\lambda+\mu_1)S_e\\ 0\\ 0\\ 0\\ \Lambda_2-\mu_1L\end{pmatrix} \tag{30}$$

Solving for $\frac{dS_u}{dt}=\ \Lambda_1-(e_0+\mu_1)S_u$ Using integrating factor method we have

$$S_u(t)=\frac{\Lambda_1}{e_0+\mu_1}+\left[S_u(0)-\frac{\Lambda_1}{e_0+\mu_1}\right]e^{-(e_0+\mu_1)t}$$

As $t\to\infty, S_u(t)\to\frac{\Lambda_1}{e_0+\mu_1}$

Thus, the following solutions were also obtained:

$$S_e(t)=\frac{e_0S_u}{(\beta_2\lambda+\mu_1)}+\left[S_e(0)-\frac{e_0S_u}{(\beta_2\lambda+\mu_1)}\right]e^{-((\beta_2\lambda+\mu_1)t}$$

$$t\to\infty, S_e\to\frac{e_0S_u}{(\beta_2\lambda+\mu_1)}$$

$$L(t)=\frac{\Lambda_2}{\mu_1}+\left[L(0)-\frac{\Lambda_2}{\mu_1}e^{\mu_1t}\right]e^{-\mu_1t}$$

As $t\to\infty, L(t)\to\frac{\Lambda_2}{\mu_1}$

Therefore, condition (a) above is satisfied.

$$G(X,Z)=\begin{pmatrix}(\psi_1I_1+\psi_2I_2)(1-\omega)S_u+(1-\theta)\lambda S_e+\psi_3LI_1+\alpha_3R-(\alpha_1+\mu_1)E\\ (1-\rho)\alpha_1E-(\rho_1+\rho_2+\rho_3+\psi_3L+\mu_1)I_1\\ \rho\alpha_1E+\rho_1\sigma_1I_1-(\psi_4L+\sigma_2+\mu_1+\mu_2)I_2\end{pmatrix} \tag{31}$$

$$P=D_zG(X,0) \tag{32}$$

$$G(X,Z)=PZ-G(X,Z) \tag{33}$$

where

$$PZ=\begin{pmatrix}-(\alpha_1+\mu_1)E+(\psi_1\beta_1S_u+\psi_3L)I_1+\psi_2\beta_1S_uI_2\\ (1-\rho)\alpha_1E+-(\rho_1+\rho_2+\rho_3+\psi_3L+\mu_1)I_1\\ \rho\alpha_1E+\rho_1\sigma_1I_1+(\psi_4L+\sigma_2+\mu_1+\mu_2)I_2\end{pmatrix} \tag{34}$$

Thus,

$$G(X,Z) = PZ - G(X,Z)$$

$$PZ - G(X,Z) = \begin{pmatrix} -(\alpha_1+\mu_1)E + (\psi_1\beta_1 S_u + \psi_3 L)I_1 + \psi_2\beta_1 S_u I_2 \\ (1-\rho)\alpha_1 E + -(\rho_1+\rho_2+\rho_3+\psi_3 L+\mu_1)I_1 \\ \rho\alpha_1 E + \rho_1\sigma_1 I_1 + \left(\psi_4 L + \sigma_2 + \mu_1 + \mu_2\right)I_2 \end{pmatrix} - \begin{pmatrix} (\psi_1 I_1 + \psi_2 I_2)(1-\omega)S_u + (1-\theta)\lambda S_e + \psi_3 L I_1 + \alpha_3 R - \left(\alpha_1+\mu_1\right)E \\ (1-\rho)\alpha_1 E - (\rho_1+\rho_2+\rho_3+\psi_3 L+\mu_1)I_1 \\ \rho\alpha_1 E + \rho_1\sigma_1 I_1 - (\psi_4 L + \sigma_2 + \mu_1 + \mu_2)I_2 \end{pmatrix}$$

$$= \begin{pmatrix} -(\alpha_1+\mu_1)E + (\psi_1\beta_1 S_u + \psi_3 L)I_1 + \psi_2\beta_1 S_u I_2 - (\psi_1 I_1 + \psi_2 I_2)(1-\omega)S_u - (1-\theta)\lambda S_e - \psi_3 L I_1 - \alpha_3 R + \left(\alpha_1+\mu_1\right)E \\ (1-\rho)\alpha_1 E + -(\rho_1+\rho_2+\rho_3+\psi_3 L+\mu_1)I_1 - (1-\rho)\alpha_1 E + (\rho_1+\rho_2+\rho_3+\psi_3 L+\mu_1)I_1 \\ \rho\alpha_1 E + \rho_1\sigma_1 I_1 + \left(\psi_4 L + \sigma_2 + \mu_1 + \mu_2\right)I_2 - \rho\alpha_1 E - \rho_1\sigma_1 I_1 - (\psi_4 L + \sigma_2 + \mu_1 + \mu_2)I_2 \end{pmatrix} \tag{35}$$

$$G(X,Z) = \begin{pmatrix} G_1(X,Z) \\ G_2(X,Z) \\ G_3(X,Z) \end{pmatrix} = \begin{pmatrix} \dfrac{\psi_1\beta_1\Lambda_1 I_1 + \psi_2\beta_1\Lambda_1 I_2}{e_0 + \mu_1} - \left[\left(\psi_1\Lambda_1 I_1 + \psi_2\Lambda_1 I_2\right)\beta_1 S_u + \alpha_3 R\right] \\ 0 \\ 0 \end{pmatrix} \tag{36}$$

Then when,

$\dfrac{\psi_1\beta_1\Lambda_1 I_1 + \psi_2\beta_1\Lambda_1 I_2}{e_0 + \mu_1} > \left[\left(\psi_1\Lambda_1 I_1 + \psi_2\Lambda_1 I_2\right)\beta_1 S_u + \alpha_3 R\right]$ it implies $G_1(X,Z) > 0$

$G_2(X,Z), G_3(X,Z) = 0$.

Therefore, the two conditions (a) and (b) above are satisfied. Hence the system is Globally and Asymptotically Stable (G.A.S).

3.7 Sensitivity Analysis of Drug Abuse Model

In this section, we investigated the relevance of each transmission parameter of the drug abuse model (1) on the dynamics of drug abuse. The R_0 parameter plays a determinant role in the model, this was to identify parameters that influences basic reproduction R_0 and, in turn, the transmission of the drug abuse. The sensitivity index was computed using the forward sensitivity index formula:

$$\gamma_P^{R_0} = \frac{\partial R_0}{\partial P} \times \frac{P}{R_0} \tag{37}$$

where:

R_0 Is the basic reproduction number

P is the parameter of interest

Solving for the parameters of interest, we obtained the results as presented in table 3

Table 3: Parameter values for R_0 used in the model and their sensitivity indices

Parameter	Index	Baseline Value	Source	Sensitivity Index
ρ	+	0.2	Motanya and Kuznetsov (2021)	0.0519

ρ_1	+	0.4	Motanya and Kuznetsov (2021)	0.0025
α_1	-	0.039	Motanya and Kuznetsov (2021)	-0.6178
σ_1	+	0.0301	Motanya and Kuznetsov (2021)	0.0025
σ_2	+	0.35	Motanya and Kuznetsov (2021))	0.769
ψ_1	+	0.0001	Motanya and Kuznetsov (2021)	0.945
ψ_2	+	0.0003	Motanya and Kuznetsov (O2021)	0.0550
ψ_3	-	0.0005	Assumed	-0.000706
ψ_4	+	0.0007	Assumed	0.0201
Λ_1	+	5	Motanya and Kuznetsov (2021)	1
Λ_2	+	0.03	Assumed	0.0193
μ_1	-	0.02	Motanya and Kuznetsov (2021)	-0.334
μ_2	+	0.059	Motanya and Kuznetsov (2021)	0.129
β_1	+	0.92	Assumed	1
e_0	-	0.0001	Assumed	-0.0050

The sensitivity indices results presented in Table 3 show that, those parameters with positive indices $(\rho_1,\rho_2,\rho_3,\alpha_1,\sigma_1,\sigma_2,\mu_2,\psi_1,\psi_2,\psi_4,\Lambda_1,\Lambda_2,\beta_1)$ increase the R_0 ,the parameters have great impact of expanding the disease in the population if their values are increased. Similarly, those parameters with negative sensitivity indices (ρ,μ_1,ψ_3) decrease the R_0 , the parameters have the ability of minimizing the disease in the population if their values are increased while others are kept constant.

3.8 Numerical Simulation

In this section, we numerically simulated the system of equations (1) using the values from Table 3, Table 4 and Table 5 respectively. The simulation was done on, varying enlightenment rate on sub populations over time *t*, varying the influence of law enforcement rate ψ_3 on drug users and effect of law enforcement rate ψ_4 on drug users over time. This was to examine the impact of drug users on the populace and to determine the best strategy to prevent it from spreading among individuals. Figures 2 to 4 present the results.

Table 4: Initial values of the state Variables

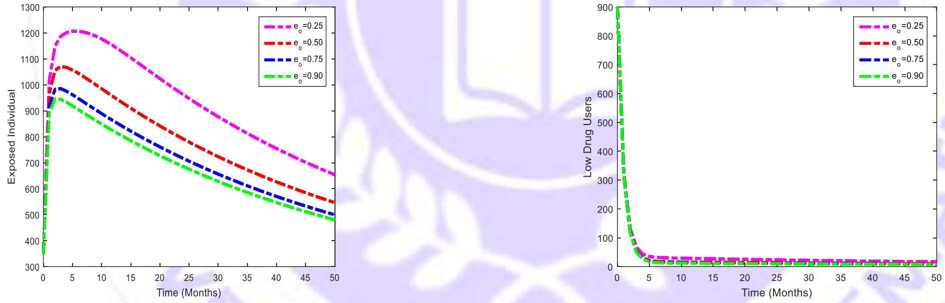
Variable	**Value**	**Source**
$S_u(0)$	10,200	Assumed
$S_e(0)$	1,500	Assumed
$E(0)$	350	Motanya and Kuznetsov (2021)

$I_1(0)$	900	Motanya and Kuznetsov (2021)
$I_2(0)$	340	Motanya and Kuznetsov (2021)
$D(0)$	150	Assumed
$Q(0)$	450	Motanya and Kuznetsov (2021)
$R(0)$	200	Motanya and Kuznetsov (2021)
$L(0)$	100	Assumed

Table 5: Parameter values

Parameter	Parameter Value	Source
θ	0.9	Assumed
λ	0.00001	Assumed
e_0	0.00001	Assumed
ω	0.08	Assumed
π	0.05	Assumed
α_2	0.007	Assumed
α_3	0.05	Assumed

3.8.1 Experiment one: Impact of enlightenment on exposed, low, heavy drug users and drug users under rehabilitation sub populations over time t



(a) (b)

Figure 2: Simulation results for varied rates of enlightenment on (a) exposed human and (b) low drug users.

(c) (d)

Figure 3: Simulation results for varied rates of enlightenment on (c) heavy drug users and (d) drug users under rehabilitation

3.8.2 Experiment two: Impact of law enforcement rate (ψ_3) on low and heavy drug users sub populations over time

(a) (b)

Figure 4: Simulation results for varied law enforcement rate ψ_3 on (a) low drug users and (b) heavy drug users

4. Results and Discussion

In this section, numerical results were discussed.

4.1 Discussion of numerical results

In this section, the numerical results found in Figures 2 to 4 were discussed as follows:

4.1.1 The impact of enlightenment on exposed, low, heavy drug users and drug users under rehabilitation sub populations over time t

In this section, we simulated the impact of enlightenment on exposed human and low drug users by Varying enlightenment parameter on the sub populations over time as shown in Figure 2(a-b), simulation to determine the impact of enlightenment on exposed human and low drug users using the system of differential equations (1) was ran. The graphs were created using the initial values and parameter values from Table 3 Table 4 and Table 5 respectively. The results indicate that the impact of enlightenment on exposed humans and low drug users demonstrate a decreasing inclination towards drug abuse and an increased likelihood of seeking assistance or adopting protective measures. This is due to higher probability of continued low drug use and an increased tendency to adopt healthy alternatives and lifestyles**.**

Here also, we simulated the impact of enlightenment on heavy drug users and drug users under rehabilitation by varying enlightenment parameter on the sub populations over time as shown in Figure 2(c-d). The results from heavy drug users and drug users under rehabilitation all indicated a steady decrease. This is due to improving health outcomes, and increased motivation to seek professional help for heavy drug users, improved treatment adherence, increased self-efficacy.

4.1.2. The impact of law enforcement rate ψ_3 on low drug users and heavy drug users sub populations over time

The simulation was conducted to analyze the impact of law enforcement agency interactions on low drug users and heavy drug users by varying the influence of law enforcement rate ψ_3 on the sub populations over time as shown in Figure 3(a-b). Figure 3(a) represents the changes in the population of Low Drug Users as the law enforcement rate varies. The simulation results demonstrate that the law enforcement rate has a minimal impact on the population of Low Drug Users. The figure shows a relatively stable population of Low Drug Users over time, regardless of the law enforcement rate. This suggests that law enforcement interventions may limit additional effects on reducing drug use among individuals who already exhibit low levels of drug abuse. Figure 3(b) showcases the effects of varying the law enforcement rate on the population of Heavy Drug Users. The simulation results indicate that as the law enforcement rate increases, the population of Heavy Drug Users decreases over time. This suggests that stricter law enforcement measures can contribute to reducing drug abuse among individuals who engage in heavy drug use. This suggests that law enforcement interventions can complement by discouraging drug use and promoting a drug endemic lifestyle among individuals in the society.

5. Conclusion

Drug abuse poses a severe threat to public health in both developed and developing countries, as seen by the high death and disability rates across the world. This study proposes a novel mathematical modeling framework for the dynamics of psychoactive drug misuse with community intervention by control agencies. The analytical results show that the drug misuse model is mathematically solid and epidemiologically well-posed in the area of concern. Both drug-abuse-free and drug-abuse-present equilibrium points were identified. The study also established that drug-free equilibrium points were locally and asymptotically stable, as well as globally asymptotically stable, when the necessary conditions were satisfied. The sensitivity index was calculated, and a sensitivity analysis was conducted. When no substantial adjustments are utilized to mitigate the problem, numerical simulations of the model equation were performed and the results described. This study on drug abuse dynamics has improved our understanding of the complexities of drug abuse, provided evidence-based treatment alternatives, and helped to reduce the impacts of substance addiction on individuals, society, and public health.

6. Recommendations

The following recommendations were made:

i. Early enlightenment should be intensified to curtail the spread of drug abuse. .
ii. Relevant authorities from the moral point of view should preach against drug abuse.
iii. Making more policies, enactment and enforcement of more laws that will guide the use of hard drugs.
iv. Banning of some selling points of drugs. In this instance, some of the recreational centres and joints that are known for selling unlawful substances should be banned and ensure that some hideout for the consumption of this illegal substances are raided.
v. Using law enforcement agencies, war against drug abuse should be intensified.

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