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Quasi Metric Space with Binary Operation and Fixed Point Theorems

Ayodele, S. Olusola[a*], Adewale, O. Kayode[a], Raji, S. Adedoyin[c], Osawaru, E. Kelly[b], Iluno Christiana[d], Oyelade, B. Eriwa[e]

[a]Tai Solarin University of Education, Ogun State, Nigeria.

[b]University of Benin, Edo State, Nigeria.

[c]Lagos State University of Education, Lagos State, Nigeria.

[d]Lagos State University of Science and Technology.

[e]Bowling Green State University, Ohio, United States.

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*Corresponding author Tel:+2347030734638
Email address: 20195208005@tasued.edu.ng
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Abstract

This paper develops a novel framework termed *quasi-metric space with binary operation*, which extends the concept of a metric space with binary operation introduced by Adewale *et al.* (2025). While their construction was based on a symmetric metric structure, the present study relaxes this assumption and considers quasi-metrics that allow asymmetry in distances. This generalization enables the exploration of non-reciprocal systems and direction-sensitive mappings. We introduce operational quasi-metric spaces, establish new structural properties, and derive several fixed point theorems for contractive-type mappings. Illustrative examples demonstrate the validity and generality of the results. Applications to convergence analysis and stability in computational systems are also outlined. Our findings generalise many known results and extend the theory of metric-type fixed points into the asymmetric domain.

1 Introduction

Metric spaces are fundamental in topology, analysis, and applied mathematics. The classical metric $b(u, v)$ assumes symmetry, $b(u, v) = b(v, u)$, which is natural in many contexts but restrictive in others. Many real-world systems, such as directed networks, preference modeling, and cost-driven processes, exhibit directional or asymmetric behaviour where the cost of going from u to v differs from that of v to u. Such cases motivate the use of *quasi-metrics*, where symmetry is relaxed but other essential properties, such as the triangle inequality, are preserved.

Recently, Adewale *et al.* (2025) introduced the notion of a *metric space with binary operation*, merging algebraic and metric structures to study fixed point results in spaces endowed with an operation $\circledcirc$. Their work retained symmetry, making it inapplicable to directional systems. This paper extends their framework by defining and analyzing a *quasi-metric space with binary operation*. Our aim is to provide a generalised structure that supports asymmetry, continuity, and algebraic interaction under binary operations.

We show that this new setting preserves most of the fundamental properties of classical metric spaces while offering broader analytical flexibility. Futhermore, we establish fixed point theorems that extend and generalise the classical results of Banach, Kannan, and Chatterjea within a quasi-metric operational environment. For further details, consult Refs.(Adewale *et al.*,2019,2024,2025; Ayodele *et al.*,2024; Smith, 2010; Johnson, 2015; Banach, 1922; Frechet, 1906;Johnson, 2015;Loyinmi, 2025; Rusu, 2009; Kirk & Sims, 2003; Berinde, 2007;Smith, 2010; Wilson, 1931; Mathew, 1994; Khamsi

& Kirk, 2001).

2 Preliminaries

We recall essential notions that will be used throughout this paper.

Definition 2.1. *A function* $b : X \times X \to [0, \infty)$ *on a nonempty set* X *is called a* quasi-metric *if for all* $u, v, w \in X$*:*

(i) $b(u, v) = 0 \Leftrightarrow u = v$,

(ii) $b(u, v) \geq 0$,

(iii) $b(u, w) \leq b(u, v) + b(v, w)$.

If $b(u, v) = b(v, u)$ *for all* $u, v \in X$*, then* b *is a metric.*

Definition 2.2. *A quasi-metric space* (X, b) *is* complete *if every Cauchy sequence* $\{u_n\}$ *with respect to* b *converges to some* $u \in X$ *such that* $b(u_n, u) \to 0$.

Definition 2.3. *A binary operation* $\circledcirc : X \times X \to X$ *is said to be* compatible *with* b *if there exists* $\lambda > 0$ *such that*

$$b(u \circledcirc v, u \circledcirc w) \leq \lambda b(v, w), \quad \forall u, v, w \in X.$$

Definition 2.4. *A mapping* $T : X \to X$ *is a* contractive-type mapping *in* $(X, b, \circledcirc)$ *if there exists* $k \in [0, 1)$ *such that*

$$b(Tu, Tv) \leq k\, b(u, v), \quad \forall u, v \in X.$$

3 Methodology

The methodology adopted in this work is purely theoretical and analytical, following a classical approach in nonlinear analysis and fixed point theory. The study proceeds through a structured development of new mathematical concepts, supported by illustrative examples and culminates in the establishment of generalised fixed point results. The essential components of the methodology are outlined as follows:

3.1 Axiomatic Framework Construction

We begin by formulating a new mathematical structure termed a *quasi-metric space with binary operation.* This involves specifying axioms governing the operational quasi-metric, defining interaction rules between the metric component and the binary operation and identifying the essential algebraic elements such as identity and operational stability.

3.2 Conceptual Generalisation

Classical notions of metric spaces, quasi-metrics and binary operations are extended to a unified framework that allows asymmetry and algebraic interaction. Through precise definitions, we generalised existing constructs such as operational metric spaces and incorporate quasi-metric asymmetry, thereby broadening their applicability.

3.3 Illustrative Examples and Structural Verification

To validate the newly introduced concepts, several examples are constructed. These examples demonstrate the behaviour of the quasi-metric under different binary operations and confirm compliance with the axioms. They serve as concrete realisations of the abstract definitions and highlight the generality of the proposed framework.

3.4 Development of Auxiliary Results

Preliminary lemmas and structural properties are established to support the main theoretical results. These foundational results ensure the internal consistency of the framework and provided essential tools such as convergence behaviour, Cauchy sequence characterisation, and continuity criteria.

3.5 Formulation and Proof of Main Theorems

The central part of the methodology involves formulating and proving fixed point theorems within the newly defined operational quasi-metric setting. Various types of contractive conditions—such as Banach-type, Kannan-type, and Chatterjea-type contractions—were considered. Each theorem is established using rigorous deductive reasoning, construction of iterative sequences and the completeness properties of the space.

3.6 Extension of Classical Fixed Point Results

The proposed results extend several classical fixed point theorems from symmetric metric spaces to asymmetric, operation-based quasi-metric environments. By relaxing the symmetry assumption and incorporating binary operations, the obtained theorems generalise well-known results in fixed point theory and open pathways for further exploration.

3.7 Discussion and Application Alignment

Finally, the implications of the results are discussed in relation to nonlinear analysis and applied mathematics. In particular, the developed framework provides a robust foundation for analyzing convergence behaviours in iterative processes, modeling direction-dependent systems and studying stability phenomena in computational and dynamic settings.

This methodology ensures a coherent and logically progressive development of the theory, while maintaining mathematical rigour and broad applicability.

4 Main Results

Definition 4.1. *Let Z be a nonempty set, $\circledcirc$, a binary operation with e as its identity element, and $b : Z^2 \to \mathbb{R}^+$. b is called an operational metric if the following axioms are satisfied:*

b_1 : $b(u, v) \geq e$;

b_2 : $b(u, v) = e$ *if and only if* $u = v$;

b_3 : $b(u, v) \leq b(u, w) \circledcirc b(w, v)$ *for all* $u, v, w \in Z$.

Z together with b is called an operational metric space. Denoted by $(Z, b, \circledcirc)$

Remark 4.2. *:*

i If the binary operation $\circledcirc$ is defined by $x \circledcirc y = x + y$, the Definition 3.1 reduces to quasi metric space introduced by Wilson (1931).

ii *If the binary operation* $\circledcirc$ *is defined by* $x \circledcirc y = x \times y$*, the Definition 3.1 reduces to quasi multiplicative metric space introduced by Ahmed and Zidan (2016).*

Example 4.3. *Let* $Z = \{x \in \mathbb{N} : x \in [2,4]\}$ *and the binary operation* $\circledcirc$ *be defined by* $x \circledcirc y = x + y - 2$*. If* $b(x,y) = |x-y| + 2$*, then* b *is a quasi metric with a binary operation,* $\circledcirc$ *and* $(Z, b, \circledcirc)$ *is a quasi metric space with binary operation,* $\circledcirc$*.*

Verification

i By definition

$$|x-y| = \begin{cases} x-y, & if \quad x-y \geq 0; \\ y-x, & if \quad x-y < 0. \end{cases}$$

So, $|x-y| \geq 0$.
Since, $|x-y| \geq 0$, $|x-y| + 2 \geq 2$ for all $x \in Z$.
Hence, $b(x,y) = |x-y| + 2 \geq e = 2$.

ii If $x \circledcirc e = x$, then $x + e - 2 = x \implies e = 2$.
$b(x,y) = e \implies |x-y| + 2 = e \implies |x-y| = 0 \implies x = y$.
Conversely,
If $x = y$, then $x - y = 0 \implies |x-y| = 0 \implies |x-y| + 2 = e \implies b(x,y) = e$

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$$\begin{aligned} b(x,y) &= |x-y| + 2 & (1) \\ &= |x-a+a-y| + 2 & (2) \\ &\leq |x-a| + |a-y| + 2 & (3) \\ &< |x-a| + 2 + |a-y| + 2 & (4) \\ &= b(x,a) + b(a,y). & (5) \end{aligned}$$

Example 4.4 (Quasi-Metric Arising from an Optimization Problem on a Directed Network)**.** *Consider a directed network* $G = (V, E)$ *where each edge* $(u,v) \in E$ *is assigned a non-negative cost* $c(u,v)$ *representing the effort or* *resources required to move from node u to node v. In many real-life systems (e.g., traffic flow, data transmission, and energy distribution), it is often the case that $c(u,v) \neq c(v,u)$, reflecting the presence of direction-dependent cost.*

For any two nodes $x, y \in V$, define

$$q(x,y) = \min_{\gamma \in \Gamma(x,y)} \sum_{(u,v)\in\gamma} c(u,v),$$

where $\Gamma(x,y)$ denotes the set of all directed paths from x to y. The function q represents the optimal cost (or minimal energy) required to travel from x to y.

We now verify that q induces a quasi-metric on V:

(i) ***Non-negativity:*** *By definition, all edge costs are non-negative, hence*

$$q(x,y) \geq 0.$$

(ii) ***Identity of indiscernibles:*** *The only path from a node to itself has cost 0, so*

$$q(x,x) = 0.$$

(iii) ***Asymmetry:*** *In general, directed edges satisfy $c(u,v) \neq c(v,u)$, therefore*

$$q(x,y) \neq q(y,x)$$

in many practical situations.

(iv) ***Triangle inequality:*** *Let $x, y, z \in V$. Any optimal path from x to z through y gives*

$$q(x,z) \leq q(x,y) + q(y,z),$$

showing that the triangle inequality holds.

Thus, (V,q) forms a quasi-metric space naturally arising from the directional optimization problem. This illustrates how quasi-metrics provide an appropriate analytic framework for modeling optimization tasks on systems with asymmetric cost structures.

Example 4.5. *Let $Z = \mathbb{N}$ and the binary operation $\circledcirc$ be defined by $x \circledcirc y = \max\{x,y\}$. If $b(x,y) = |x-y|$, then b is a quasi metric with binary operation, $\circledcirc$ and $(Z, b, \circledcirc)$ is a quasi metric space with binary operation, $\circledcirc$.*

Example 4.6. *Let $Z = \mathbb{R}$ and the binary operation $\circledcirc$ be defined by $x \circledcirc y = x + y$. If $b(x, y) = |x - y|$, then b is a quasi metric with binary operation, $\circledcirc$ and $(Z, b, \circledcirc)$ is a quasi metric space with binary operation, $\circledcirc$.*

Definition 4.7. *Let $(Z, b, \circledcirc)$ be a quasi metric space with binary operation, $\circledcirc$. An open sphere centered at x with radius r in Z is defined by*

$$S_r(x) = \{a : b(x, a) < r\}.$$

Definition 4.8. *Let $(Z, b, \circledcirc)$ be a quasi metric space with binary operation, $\circledcirc$. A closed sphere centered at x with radius r in Z is defined by*

$$S_r[x] = \{a : b(x, a) \leq r\}.$$

Definition 4.9. *Let $(Z, b, \circledcirc)$ be a quasi metric space with binary operation, $\circledcirc$. A sphere centered at x with radius r in Z is defined by*

$$S(r, x) = \{a : b(x, a) = r\}.$$

Definition 4.10. *Let $(Z, b, \circledcirc)$ be a quasi metric space with binary operation, $\circledcirc$ and $\{x_n\}$, a sequence in Z. A sequence, $\{x_n\}$ R-converge to t if for $n \in \mathbb{N}$, $b(x_n, t) \to e$ as $n \to \infty$.*

Definition 4.11. *Let $(Z, b, \circledcirc)$ be a quasi metric space with binary operation, $\circledcirc$ and $\{x_n\}$, a sequence in Z. A sequence, $\{x_n\}$ L-converge to t if for $n \in \mathbb{N}$, $b(t, x_n) \to e$ as $n \to \infty$.*

Definition 4.12. *Let $(Z, b, \circledcirc)$ be a quasi metric space with binary operation, $\circledcirc$ and $\{x_n\}$, a sequence in Z. A sequence, $\{x_n\}$ converge to t if it is both R-convergent and L-convergent to t.*

Definition 4.13. *Let $(Z, b, \circledcirc)$ be a quasi metric space with binary operation, $\circledcirc$ and $\{x_n\}$, a sequence in Z. A sequence, $\{x_n\}$ in Z is said to be a Cauchy sequence if for $n, m \in \mathbb{N}$, $b(x_n, x_m) \to e$ as $n, m \to \infty$.*

Definition 4.14. *Let $(Z_1, b_1, \circledcirc)$ and $(Z_2, b_2, \circledcirc)$ be two quasi metric spaces with same binary operation, $\circledcirc$. A $f : Z_1 \to Z_2$ is said to be continuous at a point $z \in Z_1$ if for all $\epsilon > e$ the exists $\delta > e$ such that*

$$b_1(y, z) < \delta \implies b_2(f(y), f(z)) < \epsilon.$$

The function f is continuous on Z_1 if it is continuous at every point $z \in Z_1$.

Theorem 4.15. *Let $(Z, b, \circledcirc)$ be a complete quasi metric space with binary operation, $\circledcirc$, defined by $a \circledcirc b = a + b$. Suppose $f : Z \to Z$ is a map and there exists a real number k satisfying $0 \leq k < 1$ for each $a, b \in Z$ with*

$$b(fa, fb) \leq k(b(a, b)). \tag{6}$$

Then f has a unique fixed point

Proof:
Considering (6) with an arbitrary point $x_0 \in X$ and define a sequence x_n by $x_n = f^n x_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k(b(x_{n-1}, x_n)) \tag{7}$$

Suppose f satisfies condition (7), then

$$\begin{aligned} b(x_n, x_{n+1}) &= b(Tx_{n-1}, Tx_n) && (8)\\ &\leq k(b(x_{n-1}, x_n)) && (9)\\ &\leq k^2(b(x_{n-2}, x_{n-1})) && (10) \end{aligned}$$

Using this repeatedly, we obtain

$$b(x_n, x_{n+1}) \leq k^n(b(x_0, x_1)). \tag{11}$$

By using (b_3) of Definition 3.1 with $n > m$, we have

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) \circledcirc b(x_{n-1}, x_m) && (12)\\ &= b(x_n, x_{n-1}) + b(x_{n-1}, x_m) && (13)\\ &= b(x_n, x_{n-1}) + b(x_{n-1}, x_{n-2}) + ... + b(x_{m+1}, x_m) && (14) \end{aligned}$$

With (11) and (14), we obtain

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) + b(x_{n-1}, x_{n-2}) + ... + b(x_{m+1}, x_m) && (15)\\ &\leq k^{n-1}b(x_0, x_1) + k^{n-2}b(x_0, x_1) + ... + k^m b(x_0, x_1) && (16)\\ &\leq [k^{n-1} + k^{n-2} + ... + k^m]b(x_0, x_1) && (17)\\ &\leq k^n[k^{-1} + k^{-2} + ... + k^{m-n}]b(x_0, x_1) && (18)\\ &\leq \frac{k^n}{k-1} b(x_0, x_1) && (19) \end{aligned}$$

Taking the limit of $b(x_n, x_m)$ as $n \to \infty$, we have

$$\lim_{n,m \to \infty} b(x_n, x_m) \to e. \tag{20}$$

So, $\{x_n\}$ is a S-Cauchy Sequence.
By the completeness of $(Z, b, \circledcirc)$, there exists $u \in Z$ such that $\{x_n\}$ is convergent to u.
Suppose $fu \neq u$

$$b(x_n, fu) \leq k(b(x_{n-1}, u)). \tag{21}$$

Taking the limit as n $\to \infty$ and using the fact that the function is continuous in its variables, we get

$$b(u, fu) \leq k(b(u, u)). \tag{22}$$

Hence,

$$b(u, fu) \leq e. \tag{23}$$

This is a contradiction. So, $fu = u$.
To show the uniqueness, suppose $v \neq u$ is such that $fv = v$ and $fu = u$, then

$$b(fu, fv) \leq k(b(u, v)). \tag{24}$$

Since $fu = u$ and $fv = v$, we have

$$b(u, v) \leq e. \tag{25}$$

which implies that $v = u$.

Theorem 4.16. *Let $(Z, b, \circledcirc)$ be a complete quasi metric space with binary operation, $\circledcirc$, defined by $a \circledcirc b = a + b$. Suppose $f : Z \to Z$ is a map and there exists a real number k satisfying $0 \leq k < 0.5$ for each $a, b \in Z$ with*

$$b(fa, fb) \leq k[b(a, fa) + b(b, fb)]. \tag{26}$$

Then f has a unique fixed point.

Proof:
Considering (26) with an arbitrary point $x_0 \in X$ and define a sequence x_n by $x_n = f^n x_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k[b(x_{n-1}, x_n) + b(x_{n+1}, x_n)]. \tag{27}$$

(27) implies

$$b(x_n, x_{n+1}) \leq \frac{k}{1-k} b(x_{n-1}, x_n). \tag{28}$$

If $q = \frac{k}{1-k}$, then

$$b(x_n, x_{n+1}) \leq q b(x_{n-1}, x_n). \tag{29}$$

Suppose f satisfies condition (29), then

$$\begin{aligned} b(x_n, x_{n+1}) &\leq q(b(x_{n-1}, x_n)) & (30)\\ &\leq q^2(b(x_{n-2}, x_{n-1})) & (31) \end{aligned}$$

Using this repeatedly, we obtain

$$b(x_n, x_{n+1}) \leq q^n(b(x_0, x_1)). \tag{32}$$

By using (b_3) of Definition 3.1 with $n > m$, we have

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) \circledcirc b(x_{n-1}, x_m) & (33)\\ &= b(x_n, x_{n-1}) + b(x_{n-1}, x_m) & (34)\\ &= b(x_n, x_{n-1}) + b(x_{n-1}, x_{n-2}) + ... + b(x_{m+1}, x_m) & (35) \end{aligned}$$

With (32) and (35), we obtain

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) + b(x_{n-1}, x_{n-2}) + ... + b(x_{m+1}, x_m) & (36)\\ &\leq q^{n-1}b(x_0, x_1) + q^{n-2}b(x_0, x_1) + ... + q^m b(x_0, x_1) & (37)\\ &\leq [q^{n-1} + q^{n-2} + ... + q^m]b(x_0, x_1) & (38)\\ &\leq q^n[k^{-1} + q^{-2} + ... + q^{m-n}]b(x_0, x_1) & (39)\\ &\leq \frac{q^n}{q-1}b(x_0, x_1) & (40) \end{aligned}$$

Taking the limit of $b(x_n, x_m)$ as $n \to \infty$, we have

$$\lim_{n,m\to\infty} b(x_n, x_m) \to e. \tag{41}$$

So, $\{x_n\}$ is a S-Cauchy Sequence.
By the completeness of $(Z, b, \circledcirc)$, there exists $u \in Z$ such that $\{x_n\}$ is convergent to u.
Suppose $fu \neq u$

$$b(x_n, fu) \leq k[b(x_{n-1}, x_n) + b(u, fu)]. \tag{42}$$

Taking the limit as n $\to \infty$ and using the fact that the function is continuous in its variables, we get

$$b(u, fu) \leq k(b(u, fu)). \tag{43}$$

Hence,

$$b(u, fu) \leq e. \tag{44}$$

This is a contradiction. So, $fu = u$.
To show the uniqueness, suppose $v \neq u$ is such that $fv = v$ and $fu = u$, then

$$b(fu, fv) \leq 2k(b(u, v)). \tag{45}$$

Since $fu = u$ and $fv = v$, we have

$$b(u, v) \leq e. \tag{46}$$

which implies that $v = u$.

Theorem 4.17. *Let $(Z, b, \circledcirc)$ be a complete quasi metric space with binary operation, $\circledcirc$, defined by $a \circledcirc b = a + b$. Suppose $f : Z \to Z$ is a map and there exists a real number k satisfying $0 \leq k < 0.5$ for each $a, b \in Z$ with*

$$b(fa, fb) \leq k[b(a, fb) + b(b, fa)]. \tag{47}$$

Then f has a unique fixed point

Proof:
Considering (47) with an arbitrary point $x_0 \in X$ and define a sequence x_n by $x_n = f^n x_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k[b(x_{n-1}, x_{n+1}) + b(x_n, x_n)]. \tag{48}$$

(48) implies

$$b(x_n, x_{n+1}) \leq \frac{k}{1-k} b(x_{n-1}, x_n). \tag{49}$$

If $q = \frac{k}{1-k}$, then

$$b(x_n, x_{n+1}) \leq q b(x_{n-1}, x_n). \tag{50}$$

Suppose f satisfies condition (50), then

$$\begin{aligned} b(x_n, x_{n+1}) &\leq q(b(x_{n-1}, x_n)) &(51)\\ &\leq q^2(b(x_{n-2}, x_{n-1})) &(52) \end{aligned}$$

Using this repeatedly, we obtain

$$b(x_n, x_{n+1}) \leq q^n(b(x_0, x_1)). \tag{53}$$

By using (b_3) of Definition 3.1 with $n > m$, we have

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) \circledcirc b(x_{n-1}, x_m) && (54)\\ &= b(x_n, x_{n-1}) + b(x_{n-1}, x_m) && (55)\\ &= b(x_n, x_{n-1}) + b(x_{n-1}, x_{n-2}) + ... + b(x_{m+1}, x_m) && (56) \end{aligned}$$

With (53) and (56), we obtain

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) + b(x_{n-1}, x_{n-2}) + ... + b(x_{m+1}, x_m) && (57)\\ &\leq q^{n-1}b(x_0, x_1) + q^{n-2}b(x_0, x_1) + ... + q^m b(x_0, x_1) && (58)\\ &\leq [q^{n-1} + q^{n-2} + ... + q^m]b(x_0, x_1) && (59)\\ &\leq q^n[k^{-1} + q^{-2} + ... + q^{m-n}]b(x_0, x_1) && (60)\\ &\leq \frac{q^n}{q-1}b(x_0, x_1) && (61) \end{aligned}$$

Taking the limit of $b(x_n, x_m)$ as $n \to \infty$, we have

$$\lim_{n,m\to\infty} b(x_n, x_m) \to e. \tag{62}$$

So, $\{x_n\}$ is a S-Cauchy Sequence.
By the completeness of $(Z, b, \circledcirc)$, there exists $u \in Z$ such that $\{x_n\}$ is convergent to u.
Suppose $fu \neq u$

$$b(x_n, fu) \leq k[b(x_{n-1}, fu) + b(u, x_n)]. \tag{63}$$

Taking the limit as n $\to \infty$ and using the fact that the function is continuous in its variables, we get

$$b(u, fu) \leq k(b(u, fu)). \tag{64}$$

Hence,

$$b(u, fu) \leq e. \tag{65}$$

This is a contradiction. So, $fu = u$.
To show the uniqueness, suppose $v \neq u$ is such that $fv = v$ and $fu = u$, then

$$b(fu, fv) \leq 2k(b(u, v)). \tag{66}$$

Since $fu = u$ and $fv = v$, we have

$$b(u, v) \leq e. \tag{67}$$

which implies that $v = u$.

Theorem 4.18. *Let $(Z, b, \circledcirc)$ be a complete quasi metric space with binary operation, $\circledcirc$, defined by $a \circledcirc b = \max\{a, b\}$. Suppose $f : Z \to Z$ is a map and there exists a real number k satisfying $0 \leq k < 1$ for each $a, b \in Z$ with*

$$b(fa, fb) \leq k(b(a, b)). \tag{68}$$

Then f has a unique fixed point.

Proof:
Considering (68) with an arbitrary point $x_0 \in X$ and define a sequence x_n by $x_n = f^n x_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k(b(x_{n-1}, x_n)) \tag{69}$$

Suppose f satisfies condition (69), then

$$\begin{aligned} b(x_n, x_{n+1}) &= b(Tx_{n-1}, Tx_n) && (70)\\ &\leq k(b(x_{n-1}, x_n)) && (71)\\ &\leq k^2(b(x_{n-2}, x_{n-1})) && (72) \end{aligned}$$

Using this repeatedly, we obtain

$$b(x_n, x_{n+1}) \leq k^n(b(x_0, x_1)). \tag{73}$$

By using (b_3) of Definition 3.1 with $n > m$, we have

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) \circledcirc b(x_{n-1}, x_m) && (74)\\ &= \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_m)\} && (75)\\ &= \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_{n-2}), ..., b(x_{m+1}, x_m)\} && (76) \end{aligned}$$

With (73) and (76), we obtain

$$\begin{aligned} b(x_n, x_m) &\leq \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_{n-2}), ..., b(x_{m+1}, x_m)\} && (77)\\ &\leq \max\{k^{n-1}b(x_0, x_1), k^{n-2}b(x_0, x_1), ..., k^m b(x_0, x_1)\} && (78)\\ &\leq \max\{k^{n-1}, k^{n-2}, ..., k^m\}b(x_0, x_1) && (79) \end{aligned}$$

Taking the limit of $b(x_n, x_m)$ as $n \to \infty$, we have

$$\lim_{n,m\to\infty} b(x_n, x_m) \to e. \tag{80}$$

So, $\{x_n\}$ is a S-Cauchy Sequence.
By the completeness of $(Z, b, \circledcirc)$, there exists $u \in Z$ such that $\{x_n\}$ is convergent to u.
Suppose $fu \neq u$

$$b(x_n, fu) \leq k(b(x_{n-1}, u)). \tag{81}$$

Taking the limit as n $\to \infty$ and using the fact that the function is continuous in its variables, we get

$$b(u, fu) \leq k(b(u, u)). \tag{82}$$

Hence,

$$b(u, fu) \leq e. \tag{83}$$

This is a contradiction. So, $fu = u$.
To show the uniqueness, suppose $v \neq u$ is such that $fv = v$ and $fu = u$, then

$$b(fu, fv) \leq k(b(u, v)). \tag{84}$$

Since $fu = u$ and $fv = v$, we have

$$b(u, v) \leq e. \tag{85}$$

which implies that $v = u$.

Theorem 4.19. *Let $(Z, b, \circledcirc)$ be a complete quasi metric space with binary operation, $\circledcirc$, defined by $a \circledcirc b = \max\{a, b\}$. Suppose $f : Z \to Z$ is a map and there exists a real number k satisfying $0 \leq k < 0.5$ for each $a, b \in Z$ with*

$$b(fa, fb) \leq k[b(a, fa) + b(b, fb)]. \tag{86}$$

Then f has a unique fixed point.

Proof:
Considering (86) with an arbitrary point $x_0 \in X$ and define a sequence x_n by $x_n = f^n x_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k[b(x_{n-1}, x_n) + b(x_{n+1}, x_n)]. \tag{87}$$

(87) implies

$$b(x_n, x_{n+1}) \leq \frac{k}{1-k} b(x_{n-1}, x_n). \tag{88}$$

If $q = \frac{k}{1-k}$, then

$$b(x_n, x_{n+1}) \leq q b(x_{n-1}, x_n). \tag{89}$$

Suppose f satisfies condition (89), then

$$\begin{aligned} b(x_n, x_{n+1}) &\leq q(b(x_{n-1}, x_n)) && (90)\\ &\leq q^2(b(x_{n-2}, x_{n-1})) && (91) \end{aligned}$$

Using this repeatedly, we obtain

$$b(x_n, x_{n+1}) \leq q^n(b(x_0, x_1)). \tag{92}$$

By using (b_3) of Definition 3.1 with $n > m$, we have

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) \circledcirc b(x_{n-1}, x_m) && (93)\\ &= \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_m)\} && (94)\\ &= \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_{n-2}), ..., b(x_{m+1}, x_m)\} && (95) \end{aligned}$$

With (92) and (95), we obtain

$$\begin{aligned} b(x_n, x_m) &\leq \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_{n-2}), ..., b(x_{m+1}, x_m)\} && (96)\\ &\leq \max\{q^{n-1}b(x_0, x_1), q^{n-2}b(x_0, x_1), ..., q^m b(x_0, x_1)\} && (97)\\ &\leq \max\{q^{n-1}, q^{n-2}, ..., q^m\} b(x_0, x_1) && (98) \end{aligned}$$

Taking the limit of $b(x_n, x_m)$ as $n \to \infty$, we have

$$\lim_{n,m\to\infty} b(x_n, x_m) \to e. \tag{99}$$

So, $\{x_n\}$ is a S-Cauchy Sequence.
By the completeness of $(Z, b, \circledcirc)$, there exists $u \in Z$ such that $\{x_n\}$ is convergent to u.
Suppose $fu \neq u$

$$b(x_n, fu) \leq k[b(x_{n-1}, x_n) + b(u, fu)]. \tag{100}$$

Taking the limit as n $\to \infty$ and using the fact that the function is continuous in its variables, we get

$$b(u, fu) \leq k(b(u, fu)). \tag{101}$$

Hence,

$$b(u, fu) \leq e. \tag{102}$$

This is a contradiction. So, $fu = u$.
To show the uniqueness, suppose $v \neq u$ is such that $fv = v$ and $fu = u$, then

$$b(fu, fv) \leq 2k(b(u, v)). \tag{103}$$

Since $fu = u$ and $fv = v$, we have

$$b(u, v) \leq e. \tag{104}$$

which implies that $v = u$.

Theorem 4.20. *Let $(Z, b, \circledcirc)$ be a complete quasi metric space with binary operation, $\circledcirc$, defined by $a \circledcirc b = \max\{a, b\}$. Suppose $f : Z \to Z$ is a map and there exists a real number k satisfying $0 \leq k < 0.5$ for each $a, b \in Z$ with*

$$b(fa, fb) \leq k[b(a, fb) + b(b, fa)]. \tag{105}$$

Then f has a unique fixed point.

Proof:
Considering (105) with an arbitrary point $x_0 \in X$ and define a sequence x_n by $x_n = f^n x_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k[b(x_{n-1}, x_{n+1}) + b(x_n, x_n)]. \tag{106}$$

(106) implies

$$b(x_n, x_{n+1}) \leq \frac{k}{1-k} b(x_{n-1}, x_n). \tag{107}$$

If $q = \frac{k}{1-k}$, then

$$b(x_n, x_{n+1}) \leq q b(x_{n-1}, x_n). \tag{108}$$

Suppose f satisfies condition (108), then

$$\begin{aligned} b(x_n, x_{n+1}) &\leq q(b(x_{n-1}, x_n)) \quad (109)\\ &\leq q^2(b(x_{n-2}, x_{n-1})) \quad (110) \end{aligned}$$

Using this repeatedly, we obtain

$$b(x_n, x_{n+1}) \leq q^n(b(x_0, x_1)). \tag{111}$$

By using (b_3) of Definition 3.1 with $n > m$, we have

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) \circledcirc b(x_{n-1}, x_m) && (112)\\ &= \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_m)\} && (113)\\ &= \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_{n-2}), ..., b(x_{m+1}, x_m)\} && (114)\end{aligned}$$

With (111) and (114), we obtain

$$\begin{aligned} b(x_n, x_m) &\leq \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_{n-2}), ..., b(x_{m+1}, x_m)\} && (115)\\ &\leq \max\{q^{n-1}b(x_0, x_1), q^{n-2}b(x_0, x_1), ..., q^m b(x_0, x_1)\} && (116)\\ &\leq \max\{q^{n-1}, q^{n-2}, ..., q^m\}b(x_0, x_1) && (117)\end{aligned}$$

Taking the limit of $b(x_n, x_m)$ as $n \to \infty$, we have

$$\lim_{n,m\to\infty} b(x_n, x_m) \to e. \tag{118}$$

So, $\{x_n\}$ is a S-Cauchy Sequence.
By the completeness of $(Z, b, \circledcirc)$, there exists $u \in Z$ such that $\{x_n\}$ is convergent to u.
Suppose $fu \neq u$

$$b(x_n, fu) \leq k[b(x_{n-1}, fu) + b(u, x_n)]. \tag{119}$$

Taking the limit as n $\to \infty$ and using the fact that the function is continuous in its variables, we get

$$b(u, fu) \leq k(b(u, fu)). \tag{120}$$

Hence,

$$b(u, fu) \leq e. \tag{121}$$

This is a contradiction. So, $fu = u$.
To show the uniqueness, suppose $v \neq u$ is such that $fv = v$ and $fu = u$, then

$$b(fu, fv) \leq 2k(b(u, v)). \tag{122}$$

Since $fu = u$ and $fv = v$, we have

$$b(u, v) \leq e. \tag{123}$$

which implies that $v = u$.

Conclusion

Conclusion

In this work, we explored quasi-metric spaces and their interaction with binary operations, introducing operational quasi-metric spaces as a general framework. The existence and uniqueness of fixed points under various contractive conditions were established, extending classical results to more general spaces. These results have significant implications in nonlinear analysis and applied mathematics, as they provide a flexible framework for studying fixed-point problems in spaces where symmetry and classical metric assumptions do not hold. Applications include iterative methods in optimization, convergence analysis of dynamic systems, and modeling in networks or preference-driven systems. Future research may extend these results to weaker contraction conditions, other binary operations, Computational Convergence or generalized metric spaces. “‘

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